

Twists of symmetric bundles

Ph. Cassou-Noguès, B. Erez, M.J. Taylor

5 March 2004

Contents

Introduction	3
1 General twists	12
1.a Symmetric bundles on schemes	12
1.b Clifford algebras	12
1.c Proof of Theorem 0.2	15
2 The twist of a bundle à la Fröhlich	17
2.a Tame coverings with odd ramification	17
2.b Definition of the twist: the bundle	18
2.c Algebraic preliminaries	18
2.d Definition of the twist: the form	22
2.e Proof of Theorem 0.6 (i) and examples of twists.	22
3 The étale case	26
3.a Proof of Theorem 0.4	29
4 The tame case	31
4.a Metabolic bundles and their twists	31
4.b The main lemma	34
4.c Normalisation along a divisor	35
4.d Proof of Theorems 0.6 (ii), 0.7 (i) and 0.8	36
4.e Proof of Theorem 0.7	40

5	Appendix: Hasse-Witt invariants of symmetric complexes	52
5.a	Symmetric complexes	52
5.b	Definition of the invariants	54
	Bibliography	55

Introduction

Invariants of general twists. A symmetric bundle (E, f) over a noetherian $\mathbf{Z}[\frac{1}{2}]$ -scheme Y is a vector bundle E over Y equipped with a symmetric isomorphism f between E and its Y -dual E^\vee . A symmetric bundle can also be viewed as a quadratic form on E and we write (E, q) if we take this point of view, or, if the form is clear from the context, we might even just write E . It is well known how to describe the set of all *twists* of (E, f) , that is the set of symmetric bundles which become isomorphic to (E, f) after an étale base extension. If $\mathbf{O}(E)$ denotes the orthogonal group (scheme) attached to (E, f) , this set is $H^1(Y, \mathbf{O}(E))$ (see [Mi], chapter 3 for a precise definition of this set). For α in $H^1(Y, \mathbf{O}(E))$, let E_α be the twist of E corresponding to α . Every symmetric bundle of rank n is a twist of the standard symmetric bundle $(T_n, q_n) = (O_Y^n, x_1^2 + \dots + x_n^2)$. Let $\mathbf{O}(n)$ denote the automorphism group of this symmetric bundle. We write α_E for the class of E in $H^1(Y, \mathbf{O}(n))$ ($n = \text{rank}(E)$).

Following Delzant [Dz] and Jardine [J1], for any symmetric bundle E over Y one can define a cohomological invariant, which generalizes the classical invariants of quadratic forms and which is known as the *total Hasse-Witt class*. This is a class $w_t(E)$ in the (graded) étale cohomology group $H^*(Y, \mathbf{Z}/2\mathbf{Z})$:

$$w_t(E) = 1 + w_1(E)t + w_2(E)t^2 + \dots$$

A brief review of the definitions of the Hasse-Witt invariants can be found in [CNET1], Sect. 1.e. The terms w_1 and w_2 in degrees one and two generalize the discriminant and the Hasse-Witt invariant respectively and have the following elementary description. We define δ^1 as the map induced by the determinant map

$$\delta^1 = \delta_E^1 : H^1(Y, \mathbf{O}(E)) \rightarrow H^1(Y, \mathbf{Z}/2\mathbf{Z})$$

and δ^2 as the boundary map

$$\delta^2 = \delta_E^2 : H^1(Y, \mathbf{O}(E)) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z})$$

associated to the exact sequence of étale sheaves of groups

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(E) \rightarrow \mathbf{O}(E) \rightarrow 1, \quad (0.1)$$

where $\widetilde{\mathbf{O}}(E)$ is a certain covering constructed with the help of the Clifford algebra of E (see 1.b for the precise definition). Then $w_1(E) = \delta_n^1(\alpha_E)$ and $w_2(E) = \delta_n^2(\alpha_E)$, where, $\delta_n^i = \delta_{\mathbf{1}_n}^i$.

Let us define a class $\Delta_t(\alpha)$ by the equality

$$w_t(E_\alpha) = w_t(E)\Delta_t(\alpha) .$$

We shall show the following theorem, which is a generalisation of a result of Serre [Se3] and which describes $\Delta_t(\alpha)$ up to terms of degree 3.

Theorem 0.2 *Let (E, q) be a symmetric bundle over a scheme Y and let α be an element of the cohomology set $H^1(Y, \mathbf{O}(E))$. Then*

$$i) \quad w_1(E_\alpha) = w_1(E) + \delta^1(\alpha) .$$

$$ii) \quad w_2(E_\alpha) = w_2(E) + w_1(E)\delta^1(\alpha) + \delta^2(\alpha) .$$

Our proof of this result consists of a cocycle computation in the spirit of the work of Serre and Fröhlich. It is based on an explicit formula for the cup-product of two 1-cocycles and the study of the behaviour of the above exact sequence (0.1) under change of forms.

Invariants of Fröhlich twists: the étale case. In the main body of the paper we will be interested in twists of symmetric bundles which are obtained from certain types of coverings of Y . Before giving the general definition we describe a special case first considered by Fröhlich (see [F] and also [Es-K-V]). Let G be a finite group and let X be a G -torsor over Y . Consider an orthogonal representation of G given by a symmetric bundle (E, q) over Y together with a group homomorphism $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$. The G -torsor X defines an element $c(X)$ in $H^1(Y, G)$ and so by push-forward along ρ it defines an element $\rho(X) = \rho_*(c(X))$ in $H^1(Y, \mathbf{O}(E))$. We seek a description of $E_{\rho(X)}$. Let $Tr_{X/Y}$ denote the bilinear trace form on $\pi_*(O_X)$.

Proposition 0.3 *Consider the O_Y -bundle $(E \otimes_{O_Y} \pi_*(O_X))^G$ of fixed points under G , where G acts diagonally: namely through ρ on the first component and through the given action on the second. Then*

$$E_{\rho(X)} = E_{\rho, X} := (E \otimes_{O_Y} \pi_*(O_X))^G$$

and $q_{\rho, X}$ may be thought of as the restriction to the O_Y -module $E_{\rho, X}$ of the form $|G|^{-1}(q \otimes Tr_{X/Y})$.

(See Prop. 3.1.) In comparing the invariants of E and $E_{\rho,X}$, there appear not only the Stiefel-Whitney classes $w_i(\rho)$ of ρ but also a new kind of invariant of an orthogonal representation called the spinor classes. Building on the work by Fröhlich [F], Kahn [K] and Snaith [Sn], Jardine showed that, for Y the spectrum of a field K of characteristic different from 2, there is a class

$$sp_t(\rho) = 1 + sp_1(\rho)t + sp_2(\rho)t^2 + \dots$$

called the *total spinor class*, which satisfies

$$w_t(E_{\rho,X})sp_t(\rho) = w_t(E)w_t(\rho)$$

in $H^*(K, \mathbf{Z}/2\mathbf{Z})$ and whose odd components are all trivial (see [J2]). In fact little else is known about the spinor class except in degree 2. We extend the work of Serre [Se1] and Fröhlich [F] to bundles over schemes. The proof of our result is different from that of Kahn in [K] Cor. 6.1.

Theorem 0.4 *Let (E, q) be a symmetric bundle over a scheme Y , let X be a G -torsor over Y and let $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$ be an orthogonal representation of G . Let $(E_{\rho,X}, q_{\rho,X})$ be the twist of (E, q) by ρ . Then, we have the equalities:*

$$i) \quad w_1(E_{\rho,X}) = w_1(E) + w_1(\rho) .$$

$$ii) \quad w_2(E_{\rho,X}) = w_2(E) + w_1(E)w_1(\rho) + w_2(\rho) + sp_2(\rho) .$$

In the proof of this theorem we shall make use of Thm. 0.2. This gives a new proof of Thm. 2.3 in [Es-K-V] and Thm. 0.2 in [CNET1] in the étale case. Note that our definition (Def. 3.a) of $sp_2(\rho)$ is different from that in [K], but it is known that the two definitions coincide when Y is the spectrum of a field, by a remark in [S1], p.127.

Fröhlich twists with tame ramification. In [Se2] Serre considered coverings of Riemann surfaces with odd ramification and showed how to obtain analogous formulæ, which involve expressions defined in terms of ramification data. Serre's work has been extended in [Es-K-V], [K] and more recently in [CNET1], [CNET2]. Let

$$\pi : X \rightarrow Y = X/G$$

be a covering which is tamely ramified along a divisor b with normal crossings and whose ramification indices are all odd. In [CNET1] we studied the

symmetric bundle $(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), \text{Tr}_{X/Y})$, where $\mathcal{D}_{X/Y}^{-1/2}$ is the locally free sheaf over X whose square is the inverse of the different of X/Y and $\text{Tr}_{X/Y}$ is the trace form. In this generality we obtained the same formula as Serre, *with no extra terms*. Still the ramification invariant, which makes its appearance, was not so well understood. Here we show that the square root of the inverse different bundle allows to give an explicit description of the twists by orthogonal representations coming from tame coverings and we show how to decompose the ramification invariant along characters. In our opinion this gives a better understanding of this invariant.

To be more precise, let π be as above and let ρ be an orthogonal representation of G

$$\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E)) ,$$

where (E, q) is a symmetric bundle over Y .

Definition 0.5. The *twist of (E, q) by ρ* is the symmetric bilinear form $(E_{\rho, X}, q_{\rho, X})$ on Y , where

$$E_{\rho, X} = (E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$$

is the G -fixed submodule of the G -module $E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$ and where $q_{\rho, X}$ is the form which is the restriction of $|G|^{-1}(q \otimes \text{Tr}_{X/Y})$ to the O_Y -module $E_{\rho, X}$.

It is clear that this generalizes the étale case, because in that case the inverse different is just O_X .

Theorem 0.6 *i) The twist $(E_{\rho, X}, q_{\rho, X})$ of (E, q) by ρ is a symmetric bundle over Y .*

ii) Let $\phi : Z \rightarrow Y$ be a scheme flat over Y and let $T' = Z \times_Y X$. For any orthogonal representation ρ of G in $\mathbf{O}(E)$ we have

$$(\phi^*(E))_{\rho, T'} = \phi^*(E_{\rho, X}) .$$

We wish to compute the difference $w_k(E_{\rho, X}) - w_k(E)$. We do not know how to do this directly based on Thm. 0.2 and a cocycle computation as for Thm. 0.4, although we suspect it might be possible to do so. Hence we proceed as in [CNET1] by reducing to the étale case. In order to carry out

this reduction we will use a number of functorial properties of the Hasse-Witt invariants and of the process of twisting, which are also dealt with in the paper.

Let G_2 be a 2-Sylow subgroup of G . We write $Z = X/G_2$ and we let T be the normalisation of the fiber product $T' = Z \times_Y X$. Hence we have the diagram.

$$\begin{array}{ccccc} T & \longrightarrow & T' = Z \times_Y X & \longrightarrow & X \\ & \searrow \pi_Z & \downarrow & & \downarrow \pi \\ & & Z & \xrightarrow{\phi} & Y \end{array}$$

It follows from [Es-K-V], Sect. 3.4, that the action of G on X induces a G -action on T and that π_Z identifies Z with T/G . Moreover we know from [CNET1], Thm. 2.2, that $\pi_Z : T \rightarrow Z$ is étale and hence a G -torsor (see [CEPT1], p. 291). Since the degree of the cover Z/Y is odd, the pull-back map $\phi^* : H^*(Y, \mathbf{Z}/2\mathbf{Z}) \rightarrow H^*(Z, \mathbf{Z}/2\mathbf{Z})$ is injective. Therefore we lose no information concerning the difference $w_k(E_{\rho,X}) - w_k(E)$ by considering the pull-back $\phi^*(w_k(E_{\rho,X})) - \phi^*(w_k(E))$. We now observe that firstly, we obtain a symmetric bundle over Z by considering the pull-back $(\phi^*(E), \phi^*(q))$ of (E, q) and secondly, that ρ induces an orthogonal representation $\phi^*(\rho) : G \rightarrow \Gamma(Z, \mathbf{O}(\phi^*(E)))$. Hence we are in a situation where we can use our construction of a twist and associate to $(\phi^*(E), \phi^*(q))$ the symmetric bundles $(\phi^*(E)_{\phi^*(\rho), T'}, \phi^*(q)_{\phi^*(\rho), T'})$ and $(\phi^*(E)_{\phi^*(\rho), T}, \phi^*(q)_{\phi^*(\rho), T})$. Using the good functorial properties of the Hasse-Witt invariants and of the process of twisting, the previous difference can be written

$$w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E)) .$$

Our strategy will be to express this difference as a sum of two terms, namely:

$$\left(w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E)_{\phi^*(\rho), T}) \right) + \left(w_k(\phi^*(E)_{\phi^*(\rho), T}) - w_k(\phi^*(E)) \right) .$$

Since the cover T/Z is étale the second term is known by Thm. 0.4. Therefore the heart of the problem will be to compute the first term. For w_1 the formula is as simple as one could wish, but for w_2 a new class appears. Before stating the result we introduce this new class $R(\rho, X)$.

The ramification invariant. We start by defining a divisor on Y which depends on the decomposition of ρ when restricted to the inertia groups of the

generic points of the irreducible components of the branch locus b of π . We number the irreducible components b_h of b by $1 \leq h \leq m$, we denote by ξ_h the generic point of b_h and by B_h an irreducible component of the ramification locus of π such that $\pi(B_h) = b_h$. We let I_h be the inertia group of the generic point of B_h . It follows from our hypotheses that I_h is cyclic of odd order e_h . The action of I_h on the cotangent space at the generic point of B_h is given by a character denoted by χ_h . For $0 \leq k < e_h/2$ we let $d_k^{(h)}(E)$ denote the rank over the d.v.r. O_{Y, ξ_h} of the χ_h^k -component of E_{ξ_h} considered as an $O_{Y, \xi_h}[I_h]$ -module (see section 4.e for the details). We then consider the divisor

$$R(\rho, X) = \sum_{h=1}^m d^{(h)}(E) b_h ,$$

where for $1 \leq h \leq m$ we have put

$$d^{(h)}(E) = \sum_{k=0}^{e_h/2} k d_k^{(h)}(E) .$$

We shall denote by the same symbol this divisor, the class in $Pic(Y)$ of the line bundle $O_Y(R(\rho, X))$ and its image in $H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$ under the boundary map

$$H^1(Y_{et}, \mathbf{G}_m) \rightarrow H^2(Y_{et}, \mathbf{Z}/2\mathbf{Z})$$

associated to the Kummer sequence

$$0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{G}_m \rightarrow \mathbf{G}_m \rightarrow 0$$

(note that under our assumptions we may identify μ_2 and $\mathbf{Z}/2\mathbf{Z}$).

Remark. 1) One recovers the ramification invariant in [CNET1] by considering the regular representation (see Example at the end of Sect. 4). In [CNET1] we used the notation $\rho(X/Y)$ for the ramification invariant. This seemed inappropriate here, as ρ is better used to denote orthogonal representations. (Serre had used $\omega(X/Y)$, which we changed because it reminded us too much of the canonical class...)

2) As pointed out to us by Serre, the ramification invariant can be viewed as a “half of a Woods-Hole element” (our terminology). Namely, when considering the Lefschetz-Riemann-Roch theorem one encounters expressions involving terms like $(1 - \zeta)^{-1}$, where ζ is a root of unity, which come from the

inverse of the term $\lambda_{-1}(\mathcal{N})$ (see for instance [E] Sect. 2.a, p. 126). Now, if ζ is of order e , then

$$\frac{1}{1-\zeta} = -\frac{1}{e} \sum_{k=1}^{e-1} k\zeta^k .$$

Invariants for Fröhlich twists with tame ramification. We are now in a position to state our next main result.

Theorem 0.7 *We have the following equalities:*

$$i) \quad w_1(\phi^*(E_{\rho,X})) = w_1(\phi^*(E)) + w_1(\phi^*(\rho)) .$$

ii)

$$\begin{aligned} w_2(\phi^*(E_{\rho,X})) &= w_2(\phi^*(E)) + w_1(\phi^*(E))w_1(\phi^*(\rho)) + w_2(\phi^*(\rho)) + \\ &\quad + sp_2(\phi^*(\rho)) + \phi^*(R(\rho, X)) . \end{aligned}$$

As in [CNET1] the proof of this result relies on two main ingredients: on the one hand a formula which expresses the difference between the total Hasse-Witt class of the two symmetric bundles

$$\Upsilon^{(0)} = (\phi^*(E_{\rho,X}, q_{\rho,X})) \quad \text{and} \quad \Upsilon^{(m)} = (\phi^*(E_{\rho,T}), (-1)^m \phi^*(q)_{\rho,T}))^{(-1)^m}$$

and on the other hand an explicit determinant computation. For reason of simplicity and when there is no risk of ambiguity we will just denote by $-E$ the symmetric bundle $(E, -q)$.

As shown in [CNET2] the way to understand the formula for the Hasse-Witt class is as a formula expressing the total Hasse-Witt class of a *metabolic symmetric complex* in terms of Chern classes of a lagrangian, that is a maximal totally isotropic subcomplex of this complex (see below for the terminology). More precisely, by decomposing the normalisation map $T \rightarrow T'$ into a sequence of normalisations, where we add in one irreducible component at a time, we show how to construct a lagrangian of $\Upsilon^{(0)} \perp -\Upsilon^{(m)}$ viewed as a symmetric complex concentrated in degree 0. We consider the sequence of Z -morphisms

$$T = T^{(m)} \rightarrow T^{(m-1)} \rightarrow \dots \rightarrow T^{(0)} = T' ,$$

numbered by the m components of the branch locus b of the covering and obtained by normalisation along a component of b as described in [CNET1], Sect. 3. We will prove in Sect. 4 that, for $1 \leq h \leq m$, we obtain an exact sequence of locally free O_Z -modules

$$0 \rightarrow (\phi^*(E) \otimes_{O_Z} I^{(h)})^G \rightarrow \phi^*(E)_{\rho, T^{(h)}} \oplus \phi^*(E)_{\rho, T^{(h+1)}} \rightarrow (\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G \rightarrow 0$$

and that for $0 \leq h \leq m-1$ the symmetric bundle defined as the orthogonal sum $(\phi^*(E)_{\rho, T^{(h)}}, \phi^*(q)_{\rho, T^{(h)}}) \perp (\phi^*(E)_{\rho, T^{(h+1)}}, -\phi^*(q)_{\rho, T^{(h+1)}})$ is metabolic.

For a bundle V of rank n over Y we denote by $c_i(V)$ the i -th Chern class of V as an element of $H^{2i}(Y, \mathbf{Z}/2\mathbf{Z})$, [Gr]. We define an element of $H^*(Y, \mathbf{Z}/2\mathbf{Z})$ by

$$d_t(V) = \sum_{i=0}^n (1 + (-1)^i t)^{n-i} c_i(V) t^{2i} .$$

To understand fully the next result it is necessary to work in the derived category of bounded complexes of vector bundles over a scheme. The reader is referred to [Ba1], [Ba2] and [Wa], for the basic theory and to [CNET2] for a discussion in our geometric context (see also the Appendix). The key point is that while $\Upsilon^{(0)} \perp -\Upsilon^{(m)}$ is not a metabolic symmetric bundle, it is, however, a metabolic complex in the derived category, if we view it as a symmetric complex concentrated in degree 0. Furthermore it then has for lagrangian the complex M_\bullet which is concentrated in degrees 1 and 0 with $M_0 = \oplus_{h=0}^{h=m-1} (\phi^*(E) \otimes I^{(h)})^G$ and $M_1 = \oplus_{h=1}^h \phi^*(E)_{\rho, T^{(h)}}$.

Using the generalisation of the main lemma of Sect. 4.b to derived categories we then get

$$w_t(\Upsilon^{(0)}) w_t(-\Upsilon^{(m)}) = d_t(M_0 - M_1) .$$

This formula lies behind the following result.

Theorem 0.8 *In $H^*(Z, \mathbf{Z}/2\mathbf{Z})$ the class $w_t(\phi^*(E_{\rho, X}, q_{\rho, X}))$ equals*

$$w_t(\phi^*(E)_{\rho, T}, (-1)^m \phi^*(q)_{\rho, T})^{(-1)^m} \prod_{0 \leq h \leq m-1} d_t((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)^{(-1)^h} .$$

Theorem 0.7 results from this by making the degree two terms explicit. It is quite remarkable to see how the ramification invariant comes out from this.

To end this introduction let us point out that: firstly Theorem 0.7 is new even for curves; secondly it provides a substantial strengthening of the main

result of [CNET1], since we no longer need to impose such stringent regularity conditions. Indeed, fix a subgroup H of G and let

$$\lambda : X \rightarrow V := X/H$$

denote the quotient map and

$$\gamma : V \rightarrow Y$$

the induced map, so that $\pi = \gamma \circ \lambda$. Note that V , being the quotient of a normal scheme by a finite group, is normal but not necessarily regular. Also observe that when λ is flat, then the symmetric bundle $(\gamma_*(\mathcal{D}_{V/Y}^{-1/2}), Tr_{V/Y})$ which, under this assumption, was the main object of investigation of [CNET1], provides an example of a twist of a symmetric bundle by an orthogonal representation which is both natural and explicit (see Example at the end of Sect. 4). So our previous work does become a particular case of the general study of this paper. Moreover, when V is not regular and hence λ not flat, then $(\gamma_*(\mathcal{D}_{V/Y}^{-1/2}), Tr_{V/Y})$ is not in general a symmetric bundle. In this case $(\pi_*(\mathcal{D}_{X/Y}^{-1/2})^H, Tr_{V/Y})$ provides the right substitute.

We should also indicate some other related developments. The twisting results for forms over fields play an important role in the work of Saito on the sign of the functional equation of the L-function of an orthogonal motive (see [S1]). Saito proves a p -adic version of Fröhlich's result: namely, he deals with Galois representations that do not necessarily have finite image. He uses this to prove a result analogous to the Fröhlich-Queyrut Theorem, which states that the global root number of real orthogonal characters equals one [FQ]. Saito's approach follows that of Deligne, who interpreted this reciprocity result in terms of the Stiefel-Whitney classes of the (local) characters [De]. More recently the results of [CNET1] have been used by Glass in [Gl] to relate the Galois invariants appearing there to ϵ -factors. On a different track, Saito has generalized Serre's original formula to the case of (smooth) non-finite morphisms $X \rightarrow Y$ [S2]. The conjunction of these results suggests a beautiful picture, parts of which are still hidden, in which the formulæ obtained in this paper seem to hold a special place.

1 General twists

1.a Symmetric bundles on schemes

For completeness, we recall the basic definitions concerning forms over schemes. We let Y be a scheme and we assume that 2 is invertible over Y , then the theory of symmetric bilinear forms over Y is equivalent to that of quadratic forms over Y . A *vector bundle* E on Y is a locally free \mathcal{O}_Y -module of finite rank. The dual of a vector bundle E is the vector bundle E^\vee such that, for any open subscheme Z of Y

$$E^\vee(Z) = \text{Hom}_{\mathcal{O}_Z}(E|_Z, \mathcal{O}_Z) .$$

There is a natural evaluation pairing $\langle \cdot, \cdot \rangle$ between E and E^\vee and one can identify E with the double dual $E^{\vee\vee}$ by

$$\kappa : E \cong E^{\vee\vee} ,$$

where $\langle \alpha, \kappa(u) \rangle = \langle u, \alpha \rangle$. A *symmetric bilinear form* on Y is a vector bundle E on Y equipped with a map of sheaves

$$q : E \times_Y E \rightarrow \mathcal{O}_Y ,$$

which on sections over an open subscheme restricts to a symmetric bilinear form. This defines an *adjoint* map

$$\varphi = \varphi_q : E \rightarrow E^\vee ,$$

which because of the symmetry assumption equals its transpose:

$$\varphi = \varphi^t : E \xrightarrow{\kappa} E^{\vee\vee} \xrightarrow{\varphi^\vee} E^\vee .$$

We shall say that (E, q) is *non-degenerate* (or unimodular) if the adjoint φ is an isomorphism. From now on we will call a *symmetric bundle* any vector bundle endowed with a non-degenerate quadratic form.

1.b Clifford algebras

The properties of the Clifford algebra and the Clifford group associated with a symmetric bundle will play an important role in the proof of Thm. 0.2.

Therefore we start by briefly recalling some of the basic material that we shall need in this section. Our references will be [Es-K-V], section 1.9, [Knu], Chapter 4, in the case of forms over a ring, and [F], Appendix 1, for a brief review in the case of forms over a field. In fact one has to observe that most of the definitions about Clifford algebras associated to a quadratic form over a field, or more generally over a commutative ring, can be generalised in our geometric context. Moreover, by reducing to affine neighbourhoods, we will essentially work with non-degenerate forms over noetherian, integral domains.

To any symmetric bundle (V, q) of constant rank n , one associates a sheaf of algebras $\mathcal{C}(q)$ over O_Y , of constant rank 2^n . As in the classical case one has the notion of odd and even elements of $\mathcal{C}(q)$ and hence a mod 2 grading. The Clifford group $\mathcal{C}^*(q)$ is the subgroup of homogeneous, invertible elements x in $\mathcal{C}(q)$ such that xvx^{-1} belongs to V for any v of V . The mod 2 grading induces a splitting

$$\mathcal{C}^*(q) = \mathcal{C}_+^*(q) \cup \mathcal{C}_-^*(q) .$$

Let σ be the anti-automorphism on $\mathcal{C}(q)$ induced by the identity on V so that $\sigma(v_1 \cdots v_m) = v_m \cdots v_1$. One verifies that the map N defined on $\mathcal{C}^*(q)$ by $N(x) = \sigma(x)x$ induces an homomorphism,

$$N : \mathcal{C}^*(q) \rightarrow \mathbf{G}_m .$$

We define the algebraic group scheme $\widetilde{\mathbf{O}}(q)$ as the kernel of this homomorphism. This group scheme also splits as $\widetilde{\mathbf{O}}_+(q) \cup \widetilde{\mathbf{O}}_-(q)$. Let x be in $\widetilde{\mathbf{O}}_\epsilon(q)$ with $\epsilon = \pm 1$, then we define $r_q(x)$ as the element of $\mathbf{O}(q)$

$$\begin{aligned} r_q(x) : V &\rightarrow V \\ v &\mapsto \epsilon vxv^{-1} . \end{aligned}$$

This defines a group homomorphism $r_q : \widetilde{\mathbf{O}}(q) \rightarrow \mathbf{O}(q)$. One can show that for each $x \in \widetilde{\mathbf{O}}_\epsilon(q)$ the element $r_q(x)$ belongs to $\mathbf{O}_+(q) = \mathbf{SO}(q)$ or $\mathbf{O}_-(q) = \mathbf{O}(q) \setminus \mathbf{SO}(q)$ depending on whether $\epsilon = 1$ or -1 . Where there is no risk of ambiguity, we will write r for r_q .

We then have constructed an exact sequence of étale sheaves of groups:

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(q) \rightarrow \mathbf{O}(q) \rightarrow 1 .$$

We recall that we have previously introduced

$$\delta^1 : H^1(Y, \mathbf{O}(q)) \rightarrow H^1(Y, \mathbf{Z}/2\mathbf{Z})$$

as the map induced by the determinant and

$$\delta^2 : H^1(Y, \mathbf{O}(q)) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z})$$

as the boundary map associated to the above exact sequence. We next consider an affine situation, namely $Y = \text{Spec}(R)$ where R is an integral domain. For a symmetric bundle (V, q) over O_Y , by abuse of notation, we shall write V , $\mathbf{O}(q)$, $\widetilde{\mathbf{O}}(q)$, for the corresponding module or groups of the global sections of these objects. For any invertible element a of $\mathcal{C}(q)$ we will denote by ι_a the inner automorphism of $\mathcal{C}(q)$ given by conjugation, $x \mapsto axa^{-1}$.

Proposition 1.1 *Let (E, q) and (F, f) be symmetric bundles over O_Y and let $\theta : (E, q) \rightarrow (F, f)$ be an isometry. Then*

- i) θ extends to an isomorphism $\tilde{\theta}$ from $\widetilde{\mathbf{O}}(q)$ onto $\widetilde{\mathbf{O}}(f)$ which induces the isomorphism $u \mapsto \theta u \theta^{-1}$ from $\text{Im}(r_q)$ onto $\text{Im}(r_f)$.*
- ii) Suppose that $(E, q) = (F, f)$ and that θ belongs to $\text{Im}(r_q)$. Let $t(\theta)$ denote a lift of θ to $\widetilde{\mathbf{O}}(q)$. If θ belongs to $\mathbf{O}_+(q)$ (resp. $\mathbf{O}_-(q)$), then on $\widetilde{\mathbf{O}}_\epsilon(q)$ there holds $\tilde{\theta} = \iota_{t(\theta)}$ (resp. $\tilde{\theta} = \epsilon \iota_{t(\theta)}$).*

PROOF. The universal property of the Clifford algebra implies that θ induces a graded isomorphism $\tilde{\theta} : \mathcal{C}(q) \rightarrow \mathcal{C}(f)$, which by restriction induces the required isomorphism. Moreover since it coincides with the identity on $\mathbf{Z}/2\mathbf{Z}$, it follows that $\tilde{\theta}$ induces a group isomorphism $s_\theta : \text{Im}(r_q) \rightarrow \text{Im}(r_f)$. Let $u \in \text{Im}(r_q)$, with $u \in \mathbf{O}_\epsilon(q)$ and let $a \in \widetilde{\mathbf{O}}_\epsilon(q)$ denote a lift of u . From the very definition of r we deduce that for any $y \in F$

$$s_\theta(u)(y) = \epsilon \tilde{\theta}(a) y \tilde{\theta}(a^{-1}) .$$

Since $\tilde{\theta}$ is an isomorphism of O_Y -algebras, the right hand side of this equality can be written

$$\epsilon \tilde{\theta}(a \tilde{\theta}^{-1}(y) a^{-1}) = \theta(\epsilon a \theta^{-1}(y) a^{-1}) = (\theta u \theta^{-1})(y) .$$

Hence we have proved that s_θ and $u \mapsto \theta u \theta^{-1}$ coincide on $\text{Im}(r_q)$.

Under the hypothesis of (ii) we obtain two automorphisms of $\mathcal{C}(q)$, namely $\tilde{\theta}$ and $\iota_{t(\theta)}$. Moreover, since $t(\theta) \in \widetilde{\mathbf{O}}_\epsilon(q)$, it follows from the definition of r_q that $\theta(x) = \epsilon t(\theta) x t(\theta)^{-1}$ for any x in E . If $\theta \in \mathbf{O}_+(q)$, then $\epsilon = 1$ and $\tilde{\theta}$ and $\iota_{t(\theta)}$, which coincide on E , coincide on $\mathcal{C}(q)$. If $\theta \in \mathbf{O}_-(q)$, then $\epsilon = -1$. Therefore $\tilde{\theta}$ and $\iota_{t(\theta)}$ will coincide on $\mathcal{C}_+(q)$ and will differ by a minus sign on $\mathcal{C}_-(q)$, so the result follows.

1.c Proof of Theorem 0.2

Let (E, q) be a symmetric bundle and let α be a class of the cohomology set $H^1(Y, \mathbf{O}(q))$. Since the set $H^1(Y, \mathbf{O}(q))$ classifies the isometry classes of twisted forms of (E, q) , we may consider a twisted form (E_α, q_α) whose class represents α . Finally we denote by (T_n, q_n) the standard/sum of squares form $(O_Y^n, x_1^2 + \cdots + x_n^2)$ and as usual we write $\mathbf{O}(n)$ for $\mathbf{O}(q_n)$. Since both (E, q) and (E_α, q_α) are twisted forms of (T_n, q_n) , there exists an affine covering $\mathcal{U} = (\mathcal{U}_i \rightarrow Y)$ for the étale topology and isometries

$$\varphi_i : (E_\alpha, q_\alpha) \times U_i \rightarrow (E, q) \times U_i$$

$$\psi_i : (E, q) \times U_i \rightarrow (T_n, q_n) \times U_i .$$

Therefore, following [Mi], Sect. 4, we deduce that $(\alpha_{ij}) = \varphi_i \varphi_j^{-1}$ and $(\gamma_{ij}) = \psi_i \psi_j^{-1}$ are 1-cocycles representing (E_α, q_α) in $H^1(\mathcal{U}/Y, \mathbf{O}(q))$ and (E, q) in $H^1(\mathcal{U}/Y, \mathbf{O}(n))$ respectively. By considering $(\delta_{ij}) = (\psi_i \varphi_i)(\psi_j \varphi_j)^{-1}$ we obtain a 1-cocycle representative of (E_α, q_α) in $H^1(\mathcal{U}/Y, \mathbf{O}(n))$. We observe that we can write

$$\delta_{ij} = \psi_i \varphi_i \varphi_j^{-1} \psi_j^{-1} = \psi_i \alpha_{ij} \psi_j^{-1} = \psi_i \psi_j^{-1} \psi_j \alpha_{ij} \psi_j^{-1} = \gamma_{ij}(\psi_j \alpha_{ij} \psi_j^{-1}) .$$

In order to obtain a representative of $w_1(E_\alpha)$, it suffices to take the image by the determinant map of the cocycle (δ_{ij}) . From the previous equalities we deduce that $\det(\delta_{ij}) = \det(\gamma_{ij}) \det(\alpha_{ij})$ which immediately implies that

$$w_1(E_\alpha) = w_1(E) + \delta^1(\alpha) ,$$

as required.

We now want to compare $w_2(E_\alpha)$ and $w_2(E)$. After refining (U_i) we may assume, [Mi], III.2.19, that each α_{ij} , (resp. γ_{ij}), is the image of an element of $\widetilde{\mathbf{O}}(q)(U_{ij})$, (resp. $\widetilde{\mathbf{O}}(n)(U_{ij})$) that we denote by α'_{ij} , (resp. γ'_{ij}). By abuse of notation for any $l \in \{i, j, k\}$ we will still denote by ψ_l the restriction to U_{ijk} of the isometry ψ_l . We now deduce from Pro. 3.2. (i) that each δ_{ij} is the image of the element $\delta'_{ij} = \gamma'_{ij} \tilde{\psi}_j(\alpha'_{ij})$. Therefore $w_2(E_\alpha)$ is the class of the 2-cocycle (b_{ijk}) where

$$b_{ijk} = \delta'_{jk} \delta'_{ik}{}^{-1} \delta'_{ij} \in \Gamma(U_{ijk}, \mathbf{Z}/2\mathbf{Z}) .$$

Using repeatedly the previous equalities we obtain that

$$b_{ijk} = \gamma'_{jk} \tilde{\psi}_k(\alpha'_{jk}) \tilde{\psi}_k(\alpha'_{ik}{}^{-1}) \gamma'_{ik}{}^{-1} \gamma'_{ij} \tilde{\psi}_j(\alpha'_{ij}) ,$$

that we write

$$b_{ijk} = \gamma'_{jk} \tilde{\psi}_k(\alpha'_{jk}) \tilde{\psi}_k(\alpha'^{-1}_{ik}) \gamma'^{-1}_{jk} (\gamma'_{jk} \gamma'^{-1}_{ik} \gamma'_{ij}) \tilde{\psi}_j(\alpha'_{ij}) .$$

We now observe that on the one hand $(\gamma'_{jk} \gamma'^{-1}_{ik} \gamma'_{ij}) = \pm 1$ and thus this factor commutes with every factor of the product, while on the other hand, since $(\alpha'_{jk} \alpha'^{-1}_{ik} \alpha'_{ij}) = \pm 1$, we have $\tilde{\psi}_k(\alpha'_{jk}) \tilde{\psi}_k(\alpha'^{-1}_{ik}) = (\alpha'_{jk} \alpha'^{-1}_{ik} \alpha'_{ij}) \tilde{\psi}_k(\alpha'^{-1}_{ij})$. Piecing these observations together we obtain the equality:

$$b_{ijk} = (\gamma'_{jk} \gamma'^{-1}_{ik} \gamma'_{ij}) (\alpha'_{jk} \alpha'^{-1}_{ik} \alpha'_{ij}) \gamma'_{jk} \tilde{\psi}_k(\alpha'^{-1}_{ij}) \gamma'^{-1}_{jk} \tilde{\psi}_j(\alpha'_{ij}) .$$

We consider this equality as a product of three factors namely

$$b_{ijk} = (\gamma'_{jk} \gamma'^{-1}_{ik} \gamma'_{ij}) (\alpha'_{jk} \alpha'^{-1}_{ik} \alpha'_{ij}) \epsilon_{ijk}$$

where $\epsilon_{ijk} = \gamma'_{jk} \tilde{\psi}_k(\alpha'^{-1}_{ij}) \gamma'^{-1}_{jk} \tilde{\psi}_j(\alpha'_{ij})$. The first two factors are 2-cocycles which respectively represent $w_2(E)$ and $\delta^2(\alpha)$.

We now want to simplify the expression for ϵ_{ijk} . This is achieved by considering the various possible signs of $\det(\gamma_{jk})$ and $\det(\alpha_{ij})$. We first observe that the equality $\gamma_{jk} \psi_k = \psi_j$ implies that $\tilde{\gamma}_{jk} \tilde{\psi}_k = \tilde{\psi}_j$. Moreover, with the notation of Prop. 1.1, we may write $\gamma'_{jk} \tilde{\psi}_k(x) \gamma'^{-1}_{jk} = \iota_{\gamma'_{jk}} \tilde{\psi}_k(x)$. We start by supposing that $\det(\gamma_{jk}) = 1$. Then it follows Prop. 1.1 *ii*) that $\iota_{\gamma'_{jk}} = \tilde{\gamma}_{jk}$, hence $\gamma'_{jk} \tilde{\psi}_k(x) \gamma'^{-1}_{jk} = \tilde{\gamma}_{jk} \tilde{\psi}_k(x) = \tilde{\psi}_j(x)$ and we conclude that $\epsilon_{ijk} = 1$. We now suppose that $\det(\gamma_{jk}) = -1$. First we assume that $\det(\alpha_{ij}) = 1$. Then, in this case, $\alpha'_{ij} \in \mathcal{C}_+(q)$ and therefore $\tilde{\psi}_k(\alpha'^{-1}_{ij}) \in \mathcal{C}_+(q_n)$. Using as before Prop. 1.1 *ii*), we obtain that

$$\gamma'_{jk} \tilde{\psi}_k(\alpha'^{-1}_{ij}) \gamma'^{-1}_{jk} = \tilde{\gamma}_{jk} \tilde{\psi}_k(\alpha'^{-1}_{ij}) = \tilde{\psi}_j(\alpha'^{-1}_{ij})$$

and we conclude that $\epsilon_{ijk} = 1$. We now suppose that $\det(\alpha_{ij}) = -1$ then $\tilde{\psi}_k(\alpha'^{-1}_{ij}) \in \mathcal{C}_-(q_n)$. Therefore, using once again Prop. 1.1 *ii*), we deduce that

$$\gamma'_{jk} \tilde{\psi}_k(\alpha'^{-1}_{ij}) \gamma'^{-1}_{jk} = -\tilde{\gamma}_{jk} \tilde{\psi}_k(\alpha'^{-1}_{ij}) = -\tilde{\psi}_j(\alpha'^{-1}_{ij}) .$$

It then follows from the definition that $\epsilon_{ijk} = -1$. As an immediate consequence of the study of these different cases we obtain the equality

$$\epsilon_{ijk} = (-1)^{\epsilon(\det(\alpha_{ij}))\epsilon(\det(\gamma_{jk}))} .$$

where for x in $\{\pm 1\}$ we define $\epsilon(x) \in \mathbf{Z}/2\mathbf{Z}$ by $x = (-1)^{\epsilon(x)}$. Therefore we conclude that (ϵ_{ijk}) is a 2-cocycle representative of the cup product $w_1(E)\delta^1(\alpha)$. This completes the proof of the theorem.

2 The twist of a bundle *à la* Fröhlich

The aim of this section is to prove Theorem 0.6: namely, that the twist of symmetric bundle by an orthogonal representation of the “Galois group” of a tame covering is again a symmetric bundle. It should be observed that in this section we are able to relax some of the hypotheses imposed in subsequent sections.

2.a Tame coverings with odd ramification

In what follows all the schemes will be defined over $\text{Spec}(\mathbf{Z}[\frac{1}{2}])$. We let X be a connected, projective, regular scheme which is either defined over the spectrum $\text{Spec}(\mathbf{F}_p)$ of the prime field of characteristic $p \neq 2$ or is flat over $\text{Spec}(\mathbf{Z}[\frac{1}{2}])$. We assume that X is equipped with a tame action by a finite group G , in the sense of Grothendieck-Murre. In particular the quotient

$$\pi : X \rightarrow Y = X/G$$

exists and is a torsor for G over Y outside a divisor b with normal crossings (see [Gr-M], [CEPT1], [CE] Appendix and [CEPT2] 1.2 and Appendix). We assume that π is a flat morphism of schemes and therefore that Y is regular. The different $\mathcal{D}_{X/Y}$ is defined as the annihilator of the sheaf $\Omega_{X/Y}^1$ of relative differentials of degree 1 (see [Mi] Rem. I.3.7). The reduced closed subscheme of X defined by the support of the different is the ramification locus of π , which we denote by $B = B(X/Y)$. Then $b = b(X/Y)$ is the reduced subscheme of Y defined by the image of B under π . There are decompositions

$$b(X/Y) = \coprod_{1 \leq h \leq m} b_h \quad \text{and} \quad B(X/Y) = \coprod_{h,k} B_{h,k} ,$$

where the b_h are the irreducible components of b and where for any fixed integer h between 1 and m , the $B_{h,k}$ are the irreducible components of B such that $\pi(B_{h,k}) = b_h$. The tameness assumption on the ramification implies that the branch locus $b(X/Y)$ on Y is a divisor with normal crossings. For each irreducible component b_h of b , we denote by ξ_h the generic point of b_h and by $\xi'_{h,k}$ a generic point of the component $B_{h,k}$ of the ramification locus on X . The ramification index (resp. residue class extension degree) of $\xi'_{h,k}$ over ξ_h which is independent of k will be denoted by $e(\xi'_h)$ (resp. $f(\xi'_h)$). We assume that the inertia group of each closed point of X has odd order, thus

every point has odd inertia and in particular the integers $e(\xi'_h)$ are odd. We introduce the divisor of X

$$\omega_{X/Y} = \sum_{h,k} (e(\xi'_h) - 1) B_{h,k}$$

and we define the square root of the codifferent as a vector bundle on X by setting

$$\mathcal{D}_{X/Y}^{-1/2} = \mathcal{O}_X(\omega_{X/Y}/2) .$$

Therefore we obtain a symmetric bundle on Y as defined in 1.a by considering $(\pi_*(\mathcal{D}_{X/Y}^{-1/2}), \text{Tr}_{X/Y})$, where $\text{Tr}_{X/Y}$ denotes the trace form.

2.b Definition of the twist: the bundle

Let ρ be an orthogonal representation of G taking values in the orthogonal group $\mathbf{O}(E)$ of the symmetric bundle E over Y . Under the assumptions that the action of G is tame and that the ramification indices are odd, we may consider the \mathcal{O}_Y -module

$$E_{\rho,X} := (E \otimes_{\mathcal{O}_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G .$$

Our first aim is to show that $E_{\rho,X}$ is a vector bundle over \mathcal{O}_Y . The proof will be a consequence of a number of elementary algebraic results. Later on, after giving a precise definition of the form $q_{\rho,X}$, we shall show that $(E_{\rho,X}, q_{\rho,X})$ is non-degenerate and, in the étale case, we identify this form to a twist of E by a cocycle (see Prop. 3.1).

2.c Algebraic preliminaries

For these auxiliary steps, as in the rest of this paper, we adopt the following conventions: R is an integral domain and M is a left $R[G]$ -module which we assume to be locally free and finitely generated as an R -module. We write $\sigma = \sum_{g \in G} g$.

Lemma 2.1 *If M is a projective $R[G]$ -module, then $M^G = \sigma M$. Furthermore:*

- i) M^G is a locally free R -module.*

ii) The map $m \mapsto \sigma m$ induces an isomorphism of R -modules from M_G onto M^G .

PROOF. All the above statements are clear when $M = R[G]$; they therefore hold for any finitely generated free $R[G]$ -module and are then easily seen to hold for a direct summand of a finitely generated free $R[G]$ -module.

We now consider a symmetric bundle (M, t) over R ; that is to say M is a finitely generated locally free R -module, equipped with a non-degenerate symmetric bilinear form t . We denote by φ_t the adjoint map $\varphi_t : M \rightarrow M^\vee$. We suppose further that M is now a projective $R[G]$ -module and that the pairing t is G -invariant. Under these assumptions we then can use Lemma 2.1 to define the symmetric bilinear form t^G on M^G by

$$t^G(x, y) = t(m, y)$$

where m is any arbitrary element of M such that $\sigma m = x$. Let I be the R -submodule of M generated by the set $\{(1 - g)m, m \in M, g \in G\}$. Since m is defined up to an element of I (Lemma 2.1 (ii)) and since t is G -invariant, one verifies immediately that t^G is well defined. Moreover one observes that for any x and y in M^G one has

$$| G | t^G(x, y) = t(x, y).$$

Proposition 2.2 *If M is a projective $R[G]$ -module, then (M^G, t^G) is a symmetric R -bundle.*

PROOF. From Lemma 2.1 we know that M^G is a locally free R -module. It remains to prove that φ_{t^G} is an R -module isomorphism from M^G onto $\text{Hom}_R(M^G, R)$. From the exact sequence

$$0 \rightarrow I \rightarrow M \rightarrow M_G \rightarrow 0 ,$$

we deduce the exact sequence

$$0 \rightarrow \text{Hom}_R(M_G, R) \rightarrow \text{Hom}_R(M, R) \rightarrow \text{Hom}_R(I, R) \rightarrow \text{Ext}_R(M_G, R) .$$

By Lemma 2.1 we know that M_G is isomorphic to M^G and is therefore R -projective. Hence we conclude that $\text{Ext}_R(M_G, R) = \{0\}$. We consider the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M^G & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\ & & \varphi' \downarrow & & \downarrow \varphi & & \downarrow \varphi'' & & \\ 0 & \rightarrow & \text{Hom}_R(M_G, R) & \rightarrow & \text{Hom}_R(M, R) & \rightarrow & \text{Hom}_R(I, R) & \rightarrow & 0 \end{array}$$

where $N = M/M^G$, $\varphi = \varphi_t$ and φ' is defined for $x \in M^G$ and $y \in M_G$ by

$$\varphi'(x)(y) = \varphi(x)(m) ,$$

where m denotes any representative of y in M . Once again, since m is defined up to an element of I , φ' is well-defined and moreover the first square and thus the diagram itself are both commutative. Applying the snake lemma to the previous diagram we obtain the exact sequence

$$0 \rightarrow \ker \varphi' \rightarrow \ker \varphi \rightarrow \ker \varphi'' \rightarrow \operatorname{coker} \varphi' \rightarrow \operatorname{coker} \varphi .$$

Since φ is an isomorphism, the previous sequence reduces to

$$0 \rightarrow \ker \varphi'' \rightarrow \operatorname{coker} \varphi' \rightarrow 0 .$$

Since M_G is isomorphic to M^G and φ' is injective, $\varphi'(M^G)$ is an R -submodule of $\operatorname{Hom}_R(M_G, R)$ of the same rank. Hence $\operatorname{coker} \varphi'$ is an R -torsion module. We now show that N and hence $\ker \varphi''$ is torsion free. Once again we can reduce consideration to the case where M is a free $R[G]$ -module with $\{e_1, e_2, \dots, e_n\}$ as a basis. Let $m = \sum_{1 \leq i \leq n} a_i e_i$ be an element of M , and let $d \in R \setminus \{0\}$ be such that $dm \in M^G$. Since $\{\sigma e_1, \sigma e_2, \dots, \sigma e_n\}$ is a basis of M^G as an R -module, there exist $\{b_i, 1 \leq i \leq n\}$ in R such that

$$dm = \sum_{1 \leq i \leq n} da_i e_i = \sum_{1 \leq i \leq n} b_i \sigma e_i .$$

It follows that for $1 \leq i \leq n$ we have $da_i = b_i \sigma$ and thus $b_i = dc_i$ with $c_i \in R$. Since M is torsion free we conclude that $\sum_{1 \leq i \leq n} a_i e_i = \sum_{1 \leq i \leq n} c_i \sigma e_i$ and hence that m belongs to M^G . Since $\ker \varphi''$ is torsion free and $\operatorname{coker} \varphi'$ is torsion, it follows that $\ker \varphi'' = \{0\} = \operatorname{coker} \varphi'$. We then have therefore shown that φ' is an isomorphism.

Let θ be the isomorphism $\theta : \operatorname{Hom}_R(M_G, R) \rightarrow \operatorname{Hom}_R(M^G, R)$ induced by the isomorphism from M^G onto M_G described in Lemma 2.1. We want to describe $\theta \circ \varphi'$. Let $x = \sigma m$ and $y = \sigma n$ be elements of M^G , then we have the equalities $\theta \circ \varphi'(x)(y) = \varphi'(x)(n) = t(x, n) = t(m, y) = t^G(x, y)$. We conclude that $\varphi_{t^G} = \theta \circ \varphi'$ and therefore that φ_{t^G} is an isomorphism as required.

For the sake of completeness we conclude this preparatory section by proving the following well-known result (see [McL] Cor. 3.3, p. 145 and p. 196, where the proof is carried out for the “other” action):

Lemma 2.3 *Let M and N be left $R[G]$ -modules. Assume that M and N are both projective R -modules and that either M or N is projective as an $R[G]$ -module. Then $M \otimes_R N$, endowed with the diagonal action of G , is a projective $R[G]$ -module.*

PROOF. Assume M to be $R[G]$ -projective. We must show that the functor $P \rightarrow \text{Hom}_{R[G]}(M \otimes_R N, P)$ from the category of $R[G]$ -modules into the category of R -modules is exact. To this end consider an arbitrary exact sequence of $R[G]$ -modules $0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$. Using the fact that N is projective as an R -module we obtain an exact sequence of R -modules

$$0 \rightarrow \text{Hom}_R(N, L') \rightarrow \text{Hom}_R(N, L) \rightarrow \text{Hom}_R(N, L'') \rightarrow 0 ,$$

which is in fact readily seen to be an exact sequence of $R[G]$ -modules. Using the fact that M is $R[G]$ -projective, from the previous sequence we obtain a further exact sequence of R -modules

$$\begin{aligned} 0 \rightarrow \text{Hom}_{R[G]}(M, \text{Hom}_R(N, L')) &\rightarrow \text{Hom}_{R[G]}(M, \text{Hom}_R(N, L)) \rightarrow \\ &\rightarrow \text{Hom}_{R[G]}(M, \text{Hom}_R(N, L'')) \rightarrow 0 . \end{aligned}$$

By the property of adjoint associativity of Hom and \otimes we obtain for any R -module P a natural isomorphism of R -modules

$$\begin{aligned} \psi_P : \text{Hom}_R(M \otimes_R N, P) &\simeq \text{Hom}_R(M, \text{Hom}_R(N, P)) \\ f &\mapsto (\psi_P(f) : a \mapsto (b \mapsto f(a \otimes b))) . \end{aligned}$$

Once again we check that ψ_P is in fact an isomorphism of G -modules and therefore induces an isomorphism of R -modules

$$\psi_P : \text{Hom}_{R[G]}(M \otimes_R N, P) \simeq \text{Hom}_{R[G]}(M, \text{Hom}_R(N, P)) .$$

Hence to conclude it suffices to replace the terms of the last exact sequence by $\text{Hom}_{R[G]}(M \otimes_R N, P)$ for $P = L, L', L''$.

2.d Definition of the twist: the form

Let U be any open subscheme of Y , then $R = O_Y(U)$ is an integral domain. Since the ramification of the cover is tame and π is flat we know that $\pi_*(\mathcal{D}_{X/Y}^{-1/2})(U)$ is a projective $O_Y(U)[G]$ -module, (see for instance section 4). It follows from Lemma 2.1 and Lemma 2.3 that $E_{\rho,X}(U)$ is projective and therefore locally free over $O_Y(U)$. We then have proved that $E_{\rho,X}$ is a vector bundle over O_Y . We now endow the vector bundle $E_{\rho,X}$ with the form

$$q_X = (q \otimes \text{Tr}_{X/Y})^G .$$

2.e Proof of Theorem 0.6 (i) and examples of twists.

We begin this subsection by deducing Thm. 0.6 (i) from the previous results. For any affine open subscheme U of Y , we consider the $O_Y(U)$ -symmetric bundle (M, t) with $M = (E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))(U)$ and $t = (q \otimes \text{Tr}_{X/Y})$. We deduce from Prop. 2.2 that t^G defines a non-degenerate symmetric form on $E_{\rho,X}(U)$ and this completes the proof of the first part of theorem. The second part of the theorem will be proved in Sect. 4, Lemma 4.3.

Next we give some examples of twists of symmetric bundles.

Example 2.4. Let μ denote the product form $\mu(x, y) = xy$ on $\pi_*(\mathcal{D}_{X/Y}^{-1/2})$. This is a form in a generalised sense, in that it takes its values in the ideal $\pi_*(\mathcal{D}_{X/Y}^{-1})$ instead of the ring O_Y . We want to show that $q_{\rho,X}$ is nothing else than the restriction to $E_{\rho,X} = (E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ of the form $t_X := q \otimes \mu$ on $(E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))$.

Proposition 2.5

$$t_X(x, y) = q_{\rho,X}(x, y) .$$

PROOF. In order to prove the equality it suffices to prove that t_X and $q_{\rho,X}$ coincide on $E_{\rho,X}(U)$ for any affine open U in Y . Moreover any element of

$E_{\rho,X}(U)$ can be obtained as a sum of elements of the form $\sigma(m \otimes n)$. So let $x = \sigma(m \otimes n)$ and $y = \sigma(m' \otimes n')$. We have the equalities

$$\begin{aligned} t_X(x, y) &= q_{\rho,X}(\sum_{g \in G} (\rho(g)m \otimes gn), \sum_{h \in G} \rho(h)m' \otimes hn') \\ &= \sum_{g, h \in G} q(\rho(g)m, \rho(h)m')(gn)(hn') . \end{aligned}$$

Using the invariance of q by G this can be written

$$t_X(x, y) = \sum_{g \in G} q(m, \rho(g)m' \text{Tr}_{X/Y}(n(gn')))) .$$

We now observe that the right hand side of this equality can be expressed as

$$(q \otimes \text{Tr}_{X/Y})(m \otimes n, \sigma(m' \otimes n')) .$$

Finally it follows from the very definition of $q_{\rho,X}$ that this last quantity is equal to $q_{\rho,X}(x, y)$. Hence we have proved that $t_X(x, y) = q_{\rho,X}(x, y)$.

Example 2.6. Subquotients. Here we show that the square root of the inverse different appears as the twist of the standard form on a permutation module by the natural orthogonal representation attached to a Galois covering. This is to be compared with the interpretation of the trace form of an étale algebra as a twist of the standard form, which lies at the foundation of the original results by Serre and Fröhlich. We keep the hypotheses and the notation of the general set-up as introduced in this section in 2.a. We fix a subgroup H of G and we let

$$\lambda : X \rightarrow V := X/H$$

denote the quotient map and

$$\gamma : V \rightarrow Y$$

be the induced map, so that $\pi = \gamma \circ \lambda$. We note that V being the quotient of a normal scheme by a finite group is normal but not necessarily regular. Let a run over a left transversal of H in G and let E be the free bundle $\mathcal{O}_Y[G/H]$ which has for basis the left cosets aH of H in G . We consider the symmetric \mathcal{O}_Y -bundle (E, q) where q is the quadratic form on E which has $\{aH\}$ as an orthonormal basis. From now on we denote by \bar{a} the coset aH .

The left action of G by permuting the cosets $\{\bar{a}\}$ extends to an orthogonal representation

$$\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q)) .$$

Our goal is to describe the twist of (E, q) by the permutation representation ρ .

Proposition 2.7 *i) The twist of (E, q) by ρ is the symmetric bundle $(\pi_*(\mathcal{D}_{X/Y}^{-1/2})^H, \text{Tr}_{V/Y})$ on Y .*

ii) Suppose that λ is flat, then $(\pi_(\mathcal{D}_{X/Y}^{-1/2})^H, \text{Tr}_{V/Y})$ is the symmetric bundle $(\gamma_*(\mathcal{D}_{V/Y}^{-1/2}), \text{Tr}_{V/Y})$.*

PROOF. (i) We can reduce to an affine situation. For any point y of Y we denote by U a sufficiently small open affine neighbourhood of y . Since the quadratic form q is non-degenerate, the adjoint map induces an isomorphism of $\mathcal{O}_Y(U)G$ -modules $E(U) \simeq \text{Hom}_{\mathcal{O}_Y(U)}(E(U), \mathcal{O}_Y(U))$. Tensoring by $\mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U))$ and using [McL], V, Proposition 4.2, we obtain a further isomorphism

$$E(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)) \simeq \text{Hom}_{\mathcal{O}_Y(U)}(E(U), \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))$$

that we denote by α . We let G act diagonally on $E(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U))$ and by conjugation on $\text{Hom}_{\mathcal{O}_Y(U)}(E(U), \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))$:

$$(g \cdot f)(u) = g(f(\rho(g^{-1})u)), \quad u \in E(U), \quad g \in G .$$

We observe that α respects the action of G . Therefore, by taking the fixed points of both sides, it induces by restriction an isomorphism:

$$E_{\rho, X}(U) \simeq \text{Hom}_{\mathcal{O}_Y(U)[G]}(E(U), \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U))) .$$

We now consider the evaluation map

$$\text{Hom}_{\mathcal{O}_Y(U)[G]}(E(U), \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U))) \rightarrow \mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U))$$

$$f \mapsto f(\bar{1}) .$$

For any $h \in H$ one has the equalities $h(f(\bar{1})) = f(\bar{h}) = f(\bar{1})$. Hence we deduce that this map takes its values in $(\mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))^H$. Moreover we

check easily that this is an isomorphism. In summary we have constructed an isomorphism of $O_Y(U)$ -modules $\theta : E_{\rho,X}(U) \rightarrow (\mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))^H$. Using Lemma 2.1, we can now describe this isomorphism as follows. Let $x = \sigma(\bar{a} \otimes m)$ be an element of $E_{\rho,X}(U)$. The image of x by α is the module homomorphism given by $(\bar{b} \rightarrow \sum_{g \in G} q(\bar{g}a, \bar{b})gm)$. Therefore, we obtain that

$$\theta(x) = \alpha(x)(\bar{1}) = \sum_{g \in G} q(\bar{g}a, \bar{1})gm .$$

It suffices to use the fact that $\{\bar{a}, a \in G/H\}$ is an orthonormal basis to deduce that

$$\theta(x) = \theta(\sigma(\bar{a} \otimes m)) = \sum_{h \in H} (ha^{-1})m = Tr_{X/V}(a^{-1}m) .$$

We now want to show that θ transports the form $q_{\rho,X}$ to the form $Tr_{V/Y}$. Let $x = \sigma(\bar{a} \otimes m)$ and $y = \sigma(\bar{b} \otimes n)$ be elements of $E_{\rho,X}(U)$. From the very definition of $q_{\rho,X}$ we obtain the equality

$$q_{\rho,X}(x, y) = \sum_{g \in G} q(\bar{a}, \bar{g}b)Tr_{X/Y}(m(gn)) .$$

Using once again the fact that $\{\bar{a}, a \in G/H\}$ is an orthonormal basis, we deduce from the above that

$$q_{\rho,X}(x, y) = \sum_{h \in H} Tr_{X/Y}(m((ahb^{-1})n)) = \sum_{h \in H} Tr_{X/Y}(((h^{-1}a^{-1})m)(b^{-1}n)) .$$

Therefore we have proved that

$$q_{\rho,X}(x, y) = Tr_{V/Y}(\sum_{h, k \in H} h(a^{-1}m)k(b^{-1}n)) = Tr_{V/Y}(\theta(x)\theta(y)) .$$

We conclude as required that θ is an isomorphism of symmetric $O_Y(U)$ -bundles

$$(E_{\rho,X}(U), q_{\rho,X}(U)) \simeq ((\mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))^H, Tr_{V/Y}) .$$

(ii) In order to complete the proof of the proposition it suffices to show that, if λ is flat, then:

$$(\mathcal{D}_{X/Y}^{-1/2}(\pi^{-1}(U)))^H = \mathcal{D}_{V/Y}^{-1/2}(\gamma^{-1}(U)) .$$

From the transitivity formula for the differentials this last equality reduces us to showing that

$$(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U)))^H = O_V(\gamma^{-1}(U)) .$$

Since the cover $X \rightarrow V$ is tame and λ is flat, it follows that $\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U))$ is a projective $O_V(\gamma^{-1}(U))[H]$ -module. Once again from Lemma 2.1 we deduce that

$$(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U)))^H = \text{Tr}_{X/V}(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U))) .$$

From [Se4], III, Prop. 7, we obtain that

$$\text{Tr}_{X/V}(\mathcal{D}_{X/V}^{-1/2}(\pi^{-1}(U))) = O_V(\gamma^{-1}(U)) .$$

The required equality now follows.

Example 2.8. Let $\pi_1(Y)^{t,o}$ be the tame odd fundamental group of Y . Any continuous homomorphism $\rho : \pi_1(Y)^{t,o} \rightarrow \mathbf{Z}/2\mathbf{Z}$ can be seen as an orthogonal representation of a finite quotient of $\pi_1(Y)^{t,o}$ into $\Gamma(Y, \mathbf{O}(q))$ where (E, q) is the standard rank one symmetric bundle (O_Y, x^2) . Therefore, let X/Y be the étale cover defined by ρ , then the map $\rho \mapsto (E_{X,\rho}, q_{X,\rho})$ induces a canonical map

$$\text{can} : H^1(\pi_1(Y)^{t,o}, \mathbf{Z}/2\mathbf{Z}) \rightarrow H^1(Y, \mathbf{Z}/2\mathbf{Z}) .$$

3 The étale case

In this section we study the case where X is a torsor of the constant group scheme G over Y , so that X is an étale covering of Y . We consider, as earlier, a symmetric bundle (E, q) , an orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$ and the twist $(E_{\rho,X}, q_{\rho,X})$ of (E, q) by ρ which, as observed previously, coincides with the O_Y -vector bundle $(\pi_* \pi^* E)^G$.

Our aim in this section is to prove, as announced in Thm. 0.3, comparison formulæ between the first and the second Hasse-Witt invariants of (E, q) and $(E_{\rho,X}, q_{\rho,X})$ which generalise those obtained by Fröhlich, [F], Thm. 2 and Thm. 3, in the case of fields extensions.

On the one hand, following Milne [Mi], III, Proposition 4.6, we note that the isomorphism class of X , considered as a sheaf torsor for G , defines an element $c(X)$ in the cohomology set $H^1(Y, G)$. On the other hand we show that $(E_{\rho,X}, q_{\rho,X})$ is a twisted form of (E, q) ; therefore its isometry class defines an element denoted by $[E_{\rho,X}, q_{\rho,X}]$ in $H^1(Y, \mathbf{O}(q))$. Finally the morphism ρ induces a natural map $\rho_* : H^1(Y, G) \rightarrow H^1(Y, \mathbf{O}(q))$. As announced in the introduction we establish the following connection between these objects.

Proposition 3.1 *In $H^1(Y, \mathbf{O}(q))$ one has the equality*

$$\rho_*(c(X)) = [E_{\rho, X}, q_{\rho, X}] .$$

PROOF. For any sheaf of groups in the étale topology, \mathcal{F} on Y , the set $H^1(Y, \mathcal{F})$ is defined to be $\varinjlim H^1(\mathcal{U}, \mathcal{F})$ where the limit is taken over all étale coverings \mathcal{U} of Y . The strategy of the proof is to show that, for any “sufficiently fine” étale covering $\mathcal{U} = (U_i \rightarrow Y)_{i \in I}$ which trivializes X as a torsor for G , we obtain an isometry of symmetric bundles from $(E_{\rho, X} \times_Y U_i, q_{\rho, X})$ onto $(E \times_Y U_i, q)$. For such coverings \mathcal{U} we will first obtain a 1-cocycle representing $c(X)$ with values in G , then from this we obtain a 1-cocycle representing $[E_{\rho, X}, q_{\rho, X}]$ which takes values in $\mathbf{O}(q)$.

Let $\mathcal{U} = (U_i)_{i \in I}$ be a sufficiently fine, affine, étale cover which trivialises X as a G -torsor. We denote respectively by $O_Y(U_i)$ and $O_X(U_i)$ the global sections of U_i and $X \times_Y U_i$. Now $X \times_Y U_i \rightarrow U_i$ is finite and U_i affine, thus $X \times_Y U_i$ is affine. Therefore for any i the isomorphism of U_i -schemes $G_{(U_i)} \simeq X \times_Y U_i$ is induced from an $O_Y(U_i)$ - G isomorphism of algebras

$$\Phi_i : O_X(U_i) \rightarrow \text{Map}(G, O_Y(U_i)) .$$

Furthermore we note that $g_{ij} = \Phi_j \Phi_i^{-1}$ is the 1-cocycle representing $c(X)$, ([Mi], III, section 4).

We again denote by $Tr_{X/Y}$ the form induced by the trace on $O_X(U_i)$. For any elements x and y of $O_X(U_i)$, the trace $Tr_{X/Y}(xy)$ belongs to $O_Y(U_i)$. Therefore, since Φ_i is an isomorphism of $O_Y(U_i)$ -algebras, we will have

$$Tr_{X/Y}(xy) = \Phi_i(Tr_{X/Y}(xy)) = \Phi_i(Tr_{X/Y}(xy))(1) .$$

Using now that Φ_i is G -equivariant we deduce from the previous equalities that

$$Tr_{X/Y}(xy) = \sum_{g \in G} \Phi_i(x)(g) \Phi_i(y)(g) .$$

Denoting by μ_G the standard G -invariant form on $\text{Map}(G, O_Y(U_i))$ given by

$$\mu_G(f, f') = \sum_{g \in G} f(g) f'(g) ,$$

we have then proved that Φ_i is an equivariant isometry between the $O_Y(U_i)$ - G symmetric bundles $(O_X(U_i), Tr_{X/Y})$ and $(\text{Map}(G, O_Y(U_i)), \mu_G)$.

Let $(E(U_i), q)$ be the $O_Y(U_i)$ symmetric bundle defined by considering the global sections of the inverse image of (E, q) by the morphism $U_i \rightarrow Y$. After tensoring over $O_Y(U_i)$ and taking fixed points by G we deduce from Φ_i an isometry that we again denote by Φ_i

$$\Phi_i : (E_{\rho, X}(U_i), q_{\rho, X}) \rightarrow (E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)))^G, t^G),$$

where t is the form $q \otimes \mu_G$ and t^G is obtained from t by following the recipe described in Sect. 2. We now let ν_i be the morphism of $O_Y(U_i)$ -modules induced by

$$\begin{aligned} \nu_i : E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)) &\longrightarrow \text{Map}(G, E(U_i)) \\ e \otimes f &\longmapsto (g \mapsto f(g)(\rho(g)e)). \end{aligned}$$

It is easy to check that ν_i is an isomorphism (use for instance the fact that $\text{Map}(G, O_Y(U_i))$ is a free $O_Y(U_i)[G]$ -module of rank 1 with basis l , where l is defined by $l(g) = 1$ if $g = 1$ and 0 otherwise). The group G acts diagonally on the left hand side while on the right hand side it acts by $uf : g \mapsto f(gu)$. It follows from the definitions of these actions that ν_i is a G -isomorphism and thus induces an isomorphism, again denoted ν_i

$$\nu_i : (E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)))^G \rightarrow \text{Map}(G, E(U_i))^G.$$

We now observe that the map $f \mapsto f(1)$ is clearly an isomorphism from $\text{Map}(G, E(U_i))^G$ onto $E(U_i)$. Hence, finally composing this map with ν_i we have defined an isomorphism of $O_Y(U_i)$ -modules

$$\gamma_i : (E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)))^G \rightarrow E(U_i).$$

We now want to describe in some detail the map γ_i . We note that, since the set $\{gl, g \in G\}$ is a free basis of the $O_Y(U_i)$ -module $\text{Map}(G, O_Y(U_i))$, every element of $(E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)))$ can be written as a sum $\sum_{g \in G} a_g \otimes gl$ and therefore every element of the subgroup of the fixed points by G can be written as a sum $\sum_{g \in G} \rho(g)e \otimes gl$. Let x be an element of $(E(U_i) \otimes_{O_Y(U_i)} \text{Map}(G, O_Y(U_i)))^G$ written as such a sum, we then have:

$$\gamma_i(x) = \nu_i(x)(1) = \sum_{g \in G} \nu_i(\rho(g)e \otimes gl)(1) = \sum_{g \in G} (gl)(1)\rho(g)e = e.$$

We now also consider $y = \sum_{g \in G} \rho(g) e' \otimes gl$. Since $x = \sigma(e \otimes l)$ and $y = \sigma(e' \otimes l)$ it follows from the very definition of t^G , prior to Prop. 2.2, that

$$t^G(x, y) = t(x, e' \otimes l) = \sum_{g \in G} q(\rho(g)e, e') \mu_G(gl, l) .$$

Since $\{gl, g \in G\}$ is an orthonormal basis of $\text{Map}(G, O_Y)$, the right hand side of the last equality is equal to $q(e, e')$. We conclude that γ_i is an isometry and therefore that $\Phi_i^{-1} \gamma_i^{-1}$ is an isometry θ_i

$$\theta_i : (E(U_i), q) \rightarrow (E_{\rho, X}(U_i), q_{\rho, X}) .$$

We now must evaluate $\theta_i^{-1} \theta_j$ in order to obtain a 1-cocycle representing $(E_{\rho, X}, q_{\rho, X})$ as a twisted form of (E, q) . Starting with $e \in E(U_{ij})$ we obtain that

$$(\gamma_i \Phi_i \Phi_j^{-1} \gamma_j^{-1})(e) = \gamma_i \left(\sum_{g \in G} \rho(g) e \otimes (gg_{ij}^{-1})l \right) = \gamma_i \left(\sum_{u \in G} (\rho(ug_{ij})e) \otimes ul \right) = \rho(g_{ij})e .$$

This concludes the proof of the proposition.

Remark. The twist of a symmetric bundle (E, q) by an orthogonal representation ρ is always a twisted form of (E, q) and therefore is given by a class in $H^1(Y, \mathbf{O}(q))$. In the étale situation Prop. 3.1 tells us precisely that this class is the image by ρ_* of the class defining X as a torsor for G . Therefore the determination of the Hasse-Witt invariants of the twisted bundle will be obtained as an application of Theorem 0.2.

3.a Proof of Theorem 0.4

We now return to the situation considered in Prop. 3.1 and we generalise Theorems 2 and 3 in [F], from field extensions to étale covers. We start by recalling some notation and definitions. Let $\pi_1(Y)$ be the fundamental group of Y based at some chosen geometric point. We consider a representation $\rho : \pi_1(Y) \rightarrow \Gamma(Y, \mathbf{O}(q))$, where (E, q) is a symmetric Y -bundle. We assume ρ to have an open kernel N . Then N defines a finite Galois étale cover X/Y with Galois group $G = \pi_1(Y)/N$. The cohomology class $c(X)$ of $H^1(Y, G)$ defined by X , considered as a G -torsor, only depends on ρ and therefore will be denoted by $c(\rho)$. The representation ρ factorises into a homomorphism $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$ which in turn induces $\rho_* : H^1(Y, G) \rightarrow H^1(Y, \mathbf{O}(q))$ and

$\rho_{*\bar{K}} : H^1(Y, G) \rightarrow H^1(Y, \mathbf{O}(q)(\bar{K}))$ with \bar{K} denoting the separable closure of the residue field at some point. We also consider the two exact sequences

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(q) \rightarrow \mathbf{O}(q) \rightarrow 1 ,$$

and

$$1 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \widetilde{\mathbf{O}}(q)(\bar{K}) \rightarrow \mathbf{O}(q)(\bar{K}) \rightarrow 1 ,$$

where the first is an exact sequence of étale sheaves of groups and the second is the exact sequence of the \bar{K} -points of the first. These sequences induce the boundary maps

$$\delta^2 : H^1(Y, \mathbf{O}(q)) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z})$$

and

$$\delta_{\bar{K}}^2 : H^1(Y, \mathbf{O}(q)(\bar{K})) \rightarrow H^2(Y, \mathbf{Z}/2\mathbf{Z}) .$$

Definition 3.2. The *first Stiefel-Whitney class* of ρ is defined to be

$$w_1(\rho) = \delta^1(\rho_*(c(\rho))) .$$

The *second Stiefel-Whitney class* of ρ is defined by

$$w_2(\rho) = (\delta_{\bar{K}}^2 \rho_{*\bar{K}})(c(\rho)) .$$

The *spinor class* is defined to be the difference

$$sp_2(\rho) = (\delta^2 \rho_*)(c(\rho)) - (\delta_{\bar{K}}^2 \rho_{*\bar{K}})(c(\rho)) .$$

Remark. The notion of the spinor class was first introduced by Fröhlich for the case of field extensions. Fröhlich's initial definition was generalised by Khan in a geometric context. In fact it is not immediately clear that Kahn's definition coincides with ours. Nevertheless one has to observe that in the case of field extensions, when $Y = \text{Spec}(K)$ and K is a field of characteristic different from 2, then T. Saito has proved in [Sa1] Lemma 3, that the spinor class we consider here is indeed equal to Fröhlich's original one.

The proof of Thm. 0.4 is now an easy consequence of Prop. 3.1 and Thm. 0.2. Let us denote by α the cohomology class $\rho_*(c(\rho))$. It follows from Prop. 3.1 that $(E_{\rho,X}, q_{\rho,X})$ can be taken as a representative of the isometry class of twisted forms of (E, q) whose image is α . Therefore it follows from Thm. 0.2 that $w_1(E_{\rho,X}) = w_1(E) + \delta^1(\alpha)$ and $w_2(E_{\rho,X}) = w_2(E) + w_1(E)\delta^1(\alpha) + \delta^2(\alpha)$. From the above definitions we deduce that $\delta^1(\alpha) = w_1(\rho)$ and that $\delta^2(\alpha) = w_2(\rho) + sp_2(\rho)$. It now suffices to use these equalities in order to obtain Thm. 0.4.

4 The tame case

In this section we return to the general tame situation as described in Sect. 2.a. We let X be a connected projective regular scheme equipped with a tame action of the finite group G and let Y denote X/G . We assume that $\pi : X \rightarrow Y$ is flat and thus that Y is regular. We consider a symmetric Y -bundle (E, q) , an orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$ and we let $(E_{\rho, X}, q_{\rho, X})$ be the twist of (E, q) by ρ as defined previously. As in the étale case, our aim is to compare the Hasse-Witt invariants of (E, q) and $(E_{\rho, X}, q_{\rho, X})$. As previously, since $(E_{\rho, X}, q_{\rho, X})$ is a twist of (E, q) , it defines an element α in the set of cohomology classes $H^1(Y, \mathbf{O}(q))$ and we can therefore apply the results of Thm. 0.2. Nevertheless we cannot yet give a description of the class α as explicit as in the étale case. In order to obtain an explicit comparison formula, we shall therefore associate to our given tame cover an étale cover to which we can directly apply our previous results. As explained in the introduction, the difference between the formula for the tame cover we started with and that for the étale cover that we construct will be reflected in the appearance of a new class depending on the decomposition of the representation ρ when restricted to the inertia groups of the generic points of the irreducible components of the branch locus of the covering X/Y . Two technical notions will play an important role here: that of a metabolic bundle and that of normalisation along a divisor.

4.a Metabolic bundles and their twists

Hyperbolic forms are metabolic and the greater generality of the latter notion is the correct one when dealing with forms over non-affine schemes. We recall the definition.

Let E be a vector bundle over Y . A sub- \mathcal{O}_Y -module V of E is a *sub-bundle* of E if it is *locally* a direct summand, *i.e.* for any y in Y , there is an open Z containing y such that $V|_Z$ has a direct summand in $E|_Z$. If V is a sub-bundle of E , then V and the quotient E/V are both vector bundles. Let now (E, q) be a vector bundle endowed with a quadratic form. For a sub-module $i : V \subset E$ one defines an orthogonal complement, which is the sub- \mathcal{O}_Y -module V^\perp , whose sections over the open Z consist of those sections of E which are orthogonal to all sections of V over any open subset of Z .

Alternatively:

$$V^\perp = \ker(E \xrightarrow{\varphi_q} E^\vee \xrightarrow{i^\vee} V^\vee) .$$

If furthermore V is a sub-bundle of E , then i^\vee is an epimorphism. Assume now that (E, q) is a symmetric bundle, then φ_q is by definition an isomorphism and therefore we have an isomorphism

$$\alpha : E/V^\perp \cong V^\vee ,$$

E/V^\perp is locally free and V^\perp is also a sub-bundle. There also is an isomorphism

$$\beta : V^\perp \cong (E/V)^\vee .$$

(The module $(E/V)^\vee$ can be identified with the sub- \mathcal{O}_Y -module of E whose sections over $Z \subset Y$ are the linear forms $\lambda : E|_Z \rightarrow \mathcal{O}_Z$ which vanish on V , so $V^\perp = \varphi_q^{-1}((E/V)^\vee)$.)

A sub-bundle V of a symmetric bundle (E, q) is a *totally isotropic sub-bundle* (also called a *sub-lagrangian*) if $V \subset V^\perp$. If V is a sub-lagrangian of (E, q) , then V^\perp/V is a sub-bundle of E/V and the form on V^\perp/V obtained by reducing q modulo V is non-degenerate. A sub-bundle V of (E, q) is called a *lagrangian* if it is such that $V = V^\perp$. The symmetric bundle (E, q) is called *metabolic* if it contains a lagrangian. If V is a lagrangian in (E, q) , then $\text{rank}(E) = 2 \cdot \text{rank}(V)$ and V is in a sense a maximal totally isotropic sub-bundle. One can observe that V is a lagrangian in (E, q) if and only if one has a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & V & \rightarrow & E & \rightarrow & V^\vee \rightarrow 0 \\ & & id \downarrow & & \downarrow \varphi_q & & \downarrow id \\ 0 & \rightarrow & V \cong V^{\vee\vee} & \rightarrow & E^\vee & \rightarrow & V^\vee \rightarrow 0 . \end{array}$$

That is to say: a metabolic form is given by a symmetric self-dual short exact sequence (see [Kne], Chapt. 3). This notion can be generalized to symmetric complexes (see Appendix, [Ba1] and [Ba2]). Metabolic bundles are trivial in the Witt group $W(Y)$, but the converse does not in general hold (see [O] Ex. 2.10 for an example).

Example. The twist of a metabolic bundle is metabolic. Let (E, q) be the underlying symmetric Y -bundle of an orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(q))$, where X is a scheme endowed with a tame action by the finite group G and $Y = X/G$. We shall say that ρ is a metabolic representation

when (E, q) is metabolic with a G -invariant lagrangian V . Our goal is to prove that if (E, q) and ρ are metabolic then $(E_{\rho, X}, q_{\rho, X})$ is also metabolic.

Let V be a lagrangian of (E, q) . We then have an exact sequence of G -modules

$$0 \rightarrow V \rightarrow E \rightarrow V^\vee \rightarrow 0 .$$

Since the cover $\pi : X \rightarrow Y$ is tame and π is flat, $\pi_*(\mathcal{D}_{X/Y}^{-1/2})$ is a locally projective $O_Y[G]$ -module. Therefore it follows from Lemma 2.3 that the modules $V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$, $E \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$ and $V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})$ are all locally projective $O_Y[G]$ -modules. So, tensoring and taking G -fixed points, affords a new exact sequence of locally free $O_Y[G]$ -modules

$$0 \rightarrow (V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G \rightarrow E_{\rho, X} \rightarrow (V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G \rightarrow 0 ,$$

(see the proof of Prop. 4.4 for details). It is clear from the definition of $q_{\rho, X}$ that its restriction to $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ is the null form since V is a lagrangian for q . In order to prove that $E_{\rho, X}$ is metabolic with $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ as a lagrangian it suffices to show that the rank of $E_{\rho, X}$ is twice the rank of $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$. This will follow at once from the O_Y -module isomorphism

$$((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G)^\vee \simeq (V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G .$$

This last isomorphism can be proved as follows. Since $\pi_*(\mathcal{D}_{X/Y}^{-1/2})$ is unimodular it is self-dual and therefore

$$(V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2})) \simeq (V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^\vee .$$

Hence we deduce that $(V^\vee \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$ is isomorphic to

$$(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^\vee{}^G = \text{Hom}_{O_Y}((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))_G, O_Y) .$$

Since we know that $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))$ is a locally projective $O_Y[G]$ -module, it follows from Lemma 2.1 (ii) that $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))_G$ is isomorphic to $(V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G$. We then conclude that

$$\text{Hom}_{O_Y}((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))_G, O_Y) \simeq ((V \otimes_{O_Y} \pi_*(\mathcal{D}_{X/Y}^{-1/2}))^G)^\vee ,$$

as required.

4.b The main lemma

Here we state a result of [Es-K-V], Prop. 5.5, which is one of the main tools in calculating Hasse-Witt invariants. It provides a precise relationship between the Hasse-Witt invariants of a metabolic form and the Chern classes of a lagrangian. In [CNET2] we have shown the relevance of this lemma when dealing with symmetric complexes (see the Appendix). We recall that we have associated to any bundle V of finite rank over Y an element $d_t(V)$ of $H^*(Y, \mathbf{Z}/2\mathbf{Z})$ defined prior to Thm. 0.8 in the introduction.

Lemma 4.1 *Let (E, q) be a metabolic form with lagrangian V . Then*

$$w_t(E) = d_t(V) .$$

In particular

$$1 + w_1(E)t + w_2(E)t^2 = 1 + n(-1)t + \left(c_1(V) + \binom{n}{2}(-1) \cup (-1) \right) t^2 .$$

Let us now assume Y to be irreducible with generic point η . In our set-up we shall construct a number of metabolic bundles from symmetric bundles over Y which are isometric when restricted to the generic fiber of Y . The essential point here is that if (E, q_E) and (F, q_F) are defined over Y and agree on the generic fiber

$$E|_\eta = F|_\eta ,$$

then, under suitable assumptions, $(E \perp F, q_E \perp -q_F)$ is metabolic. A natural approach is then to consider the sub-sheaf \mathcal{G} of $E|_\eta = F|_\eta$ defined as

$$\mathcal{G} := \langle e - f | e \in E, f \in F \rangle ,$$

and the exact sequence

$$0 \rightarrow E \cap F \rightarrow E \perp F \rightarrow \mathcal{G} \rightarrow 0 ,$$

where the maps are obtained by restricting to $E \perp F$ the maps defined at the level of the generic fibers given by the diagonal map and the map sending (x, y) to $(x - y)/2$. Then, if \mathcal{G} is locally free, it follows that $E \cap F$ is a sub-bundle of $E \perp F$, which is a lagrangian and

$$E \perp F / E \cap F \simeq (E \cap F)^\vee \simeq \mathcal{G} .$$

Lemma 4.2 *Let (E, q_E) and (F, q_F) be non-degenerate forms which are isometric when restricted to η .*

a) $w_1(E) = w_1(F)$ and $c_1(E) = c_1(F)$.

b) *Consider the exact sequence*

$$0 \rightarrow E \cap F \rightarrow E \perp F \rightarrow \mathcal{G} \rightarrow 0 .$$

Assume that \mathcal{G} (and therefore $E \cap F$) is locally free over O_Y , and that $(E \perp F, q_E \perp - q_F)$ is metabolic with lagrangian $E \cap F = \mathcal{G}^\vee$. Then

$$w_t(E, q_E) \cdot w_t(F, -q_F) = d_t(\mathcal{G}) .$$

c) *With the assumptions and the notations of (b),*

$$w_2(E, q_E) - w_2(F, q_F) = c_1(\mathcal{G}) - c_1(E) .$$

In particular this sum belongs to the image of the Picard group modulo 2 in $H^2(Y, \mathbf{Z}/2\mathbf{Z})$.

PROOF. Part (a) of the Lemma is proved in [Es-K-V], Lemma 4.8; (b) is an immediate corollary of Lemma 4.1; and (c) can also be deduced from Lemma 4.1, (see [Es-K-V], Cor. 6.3).

Remark. The first Chern class $c_1(L)$ of a vector bundle L over Y is obtained as the image in $\text{Pic}(Y)$ of the class defined by the line bundle $\det(L)$ under the boundary map associated to the Kummer exact sequence deduced from the multiplication by 2 map on \mathbf{G}_m .

4.c Normalisation along a divisor

The process of normalisation along a branch divisor was introduced and studied in [CNET1]. Let G_2 be a 2-Sylow subgroup of G . We write $Z = X/G_2$ and we let T be the normalisation of the fiber product $T' = Z \times_Y X$. So we have the diagram

$$\begin{array}{ccccc} T & \longrightarrow & T' = Z \times_Y X & \xrightarrow{\phi'} & X \\ & & \downarrow & & \downarrow \pi \\ & & Z & \xrightarrow{\phi} & Y \end{array}$$

We have proved in [CNET1], Thm. 2.2 that T is regular and that the map π_Z is étale. In *loc. cit*, Sect. 3.3 we showed how to decompose the normalisation map $T \rightarrow T'$ into a sequence

$$T = T^{(m)} \rightarrow T^{(m-1)} \rightarrow \dots \rightarrow T^{(0)} = T' ,$$

of flat Z -covers, with each $\pi^{(h)} : T^{(h)} \rightarrow Z$ having the property that $\mathcal{D}_{T^{(h)}/Z}^{-1/2}$ is a well defined $T^{(h)}$ -vector bundle. We set $\Lambda^{(h)} = \pi_*^{(h)}(\mathcal{D}_{T^{(h)}/Z}^{-1/2})$ and define $I^{(h)} = \Lambda^{(h)} \cap \Lambda^{(h+1)}$ and $\mathcal{G}^{(h)} = (\Lambda^{(h)} + \Lambda^{(h+1)})/I^{(h)}$. Then for $0 \leq h \leq m-1$ we have short exact sequences of locally free \mathcal{O}_Z -modules

$$0 \rightarrow I^{(h)} \rightarrow \Lambda^{(h)} \oplus \Lambda^{(h+1)} \rightarrow \mathcal{G}^{(h)} \rightarrow 0 .$$

The $\Lambda^{(h)}$ all coincide on the generic fiber and we have proved that $I^{(h)}$ and $\mathcal{G}^{(h)}$ are locally free \mathcal{O}_Z -modules. We then deduced that for $1 \leq h \leq m-1$, $(\Lambda^{(h)}, (-1)^h Tr_{T^{(h)}/Z}) \perp (\Lambda^{(h+1)}, (-1)^{(h+1)} Tr_{T^{(h+1)}/Z})$ is a metabolic bundle which satisfies the hypotheses of Lemma 4.2.

4.d Proof of Theorems 0.6 (ii), 0.7 (i) and 0.8

As explained in the introduction, our strategy consists of considering $w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E))$ as a sum of two terms, namely:

$$(w_k(\phi^*(E)_{\phi^*(\rho), T'}) - w_k(\phi^*(E)_{\phi^*(\rho), T})) + (w_k(\phi^*(E)_{\phi^*(\rho), T}) - w_k(\phi^*(E))) .$$

The cover T/Z is étale and so it is a G -torsor. Hence the second term is known by Thm. 0.4 for $k = 1, 2$. Therefore the main part of this section will be devoted to computing the first term.

Our first aim is to prove that our twisting process behaves well under flat base change (see Thm. 0.6 (ii)).

Lemma 4.3 *For any orthogonal representation $\rho : G \rightarrow \Gamma(Y, \mathbf{O}(E))$*

$$\phi^*(E)_{\phi^*(\rho), T'} = \phi^*(E_{\rho, X}) .$$

PROOF. It suffices to prove that the sections of these vector bundles coincide over the basis for the topology of Z given by the $\phi^{-1}(V)$ where V runs over the affine open subsets of Y . Let $U = \phi^{-1}(V)$ be one such. Write $R = \mathcal{O}_Y(V)$, $S = \mathcal{O}_Z(U)$ and $M = E(V) \otimes_R \pi_*(\mathcal{D}_{X/Y}^{-1/2})(V)$. From the very

definitions we obtain that $\phi^*(E_{\rho,X})(U) = M^G \otimes_R S$ and $\phi^*(E)_{\phi^*(\rho),T'}(U) = (\phi^*(E)(U) \otimes_S \pi_*^{(0)}(\mathcal{D}_{T'/Z}^{-1/2})(U))^G$. We now use the fact that on the one hand $\phi^*(E)(U) = E(V) \otimes_R S$ while on the other hand by [CNET1], Lemma 3.7 we know that $\pi_*^{(0)}(\mathcal{D}_{T'/Z}^{-1/2})(U) = S \otimes_R \pi_*(\mathcal{D}_{X/Y}^{-1/2})(V)$. Therefore we are reduced to proving that $(M \otimes_R S)^G = M^G \otimes_R S$ and this follows at once since S is flat over R .

To compare the Hasse-Witt invariants of $\phi^*(E)_{\phi^*(\rho),T'}$ and $\phi^*(E)_{\phi^*(\rho),T}$ we now use the decomposition of the normalisation map

$$T = T^{(m)} \rightarrow T^{(m-1)} \rightarrow \dots \rightarrow T^{(0)} = T' ,$$

recalled in the previous sub-section, to construct a new family of metabolic bundles which allows us to use the Main Lemma. For any $0 \leq h \leq m$, the group G acts on $T^{(h)}$ and ρ induces an orthogonal representation $\rho : G \rightarrow \Gamma(Z, \mathbf{O}(\phi^*(q)))$. Therefore we can consider the twist of $(\phi^*(E), \phi^*(q))$ by this representation. For simplicity and when there is no ambiguity on the choice of the representation we will denote by $(\phi^*(E)_{T^{(h)}}, \phi^*(q)_{T^{(h)}})$ the symmetric bundle $(\phi^*(E)_{\phi^*(\rho),T^{(h)}}, \phi^*(q)_{\phi^*(\rho),T^{(h)}})$. The principal advantage in considering the decomposition of normalisation into m steps is that we will be able to compare the Hasse-Witt invariants of two consecutive terms $\phi^*(E)_{T^{(h)}}$ and $\phi^*(E)_{T^{(h+1)}}$.

For $0 \leq h \leq m-1$ we have the short exact sequence of locally free \mathcal{O}_Z -modules

$$0 \rightarrow I^{(h)} \rightarrow \Lambda^{(h)} \oplus \Lambda^{(h+1)} \rightarrow \mathcal{G}^{(h)} \rightarrow 0 .$$

Since $\phi^*(E)$ is a locally free \mathcal{O}_Z -module we obtain a new exact sequence

$$0 \rightarrow \phi^*(E) \otimes_{\mathcal{O}_Z} I^{(h)} \rightarrow \phi^*(E) \otimes_{\mathcal{O}_Z} \Lambda^{(h)} \oplus \phi^*(E) \otimes_{\mathcal{O}_Z} \Lambda^{(h+1)} \rightarrow \phi^*(E) \otimes_{\mathcal{O}_Z} \mathcal{G}^{(h)} \rightarrow 0 .$$

From the definitions of $I^{(h)}$ and $\mathcal{G}^{(h)}$ it is clear that these modules are G -modules and that the morphisms in this sequence respect the action of G when G acts diagonally on the tensor products. Next we consider the sequence obtained by taking G -fixed points.

Proposition 4.4 *For $0 \leq h \leq m-1$*

$$0 \rightarrow (\phi^*(E) \otimes_{\mathcal{O}_Z} I^{(h)})^G \rightarrow \phi^*(E)_{T^{(h)}} \oplus \phi^*(E)_{T^{(h+1)}} \rightarrow (\phi^*(E) \otimes_{\mathcal{O}_Z} \mathcal{G}^{(h)})^G \rightarrow 0$$

is an exact sequence of locally free \mathcal{O}_Z -modules.

PROOF. It will be proved in Lemma 4.6, that $I^{(h)}, \Lambda^{(h)}$ and $\mathcal{G}^{(h)}$ are locally projective $O_Z[G]$ -modules for any h , $0 \leq h \leq m-1$. Therefore, for any (closed) point z of Z we may choose a sufficiently small affine neighbourhood U of z such that $I^{(h)}(U)$, $\mathcal{G}^{(h)}(U)$, $\Lambda^{(h)}(U)$ and $\Lambda^{(h+1)}(U)$ are projective $O_Z(U)[G]$ -modules. From Lemma 2.3 it follows that the modules obtained by tensoring these modules with $\phi^*(E)(U)$ over $O_Z(U)$ are all projective $O_Z(U)[G]$ -modules. Hence, by Lemma 2.1, we know that their G -fixed points are locally free $O_Z(U)$ -modules. We now consider the exact sequence of left $O_Z(U)[G]$ -modules

$$0 \rightarrow I^{(h)}(U) \rightarrow \Lambda^{(h)}(U) \oplus \Lambda^{(h+1)}(U) \rightarrow \mathcal{G}^{(h)}(U) \rightarrow 0.$$

By tensoring with the $O_Z(U)$ -locally free module $\phi^*(E)(U)$ we obtain a new exact sequence of $O_Z(U)[G]$ -modules, where G acts diagonally,

$$\begin{aligned} 0 \rightarrow \phi^*(E)(U) \otimes_{O_Z(U)} I^{(h)}(U) \\ \rightarrow \phi^*(E)(U) \otimes_{O_Z(U)} \Lambda^{(h)}(U) \oplus \phi^*(E)(U) \otimes_{O_Z(U)} \Lambda^{(h+1)}(U) \\ \rightarrow \phi^*(E)(U) \otimes_{O_Z(U)} \mathcal{G}^{(h)}(U) \rightarrow 0. \end{aligned}$$

Taking G -fixed points affords a further exact sequence

$$\begin{aligned} 0 \rightarrow (\phi^*(E)(U) \otimes_{O_Z(U)} I^{(h)}(U))^G \rightarrow \phi^*(E)_{T^{(h)}}(U) \oplus \phi^*(E)_{T^{(h+1)}}(U) \\ \rightarrow (\phi^*(E)(U) \otimes_{O_Z(U)} \mathcal{G}^{(h)}(U))^G \rightarrow H^1(G, \phi^*(E)(U) \otimes_{O_Z(U)} I^{(h)}(U)) \rightarrow 0. \end{aligned}$$

Using Lemma 2.3 and the fact that $I^{(h)}(U)$ is a projective $O_Z(U)[G]$ -module we conclude that $\phi^*(E)(U) \otimes_{O_Z(U)} I^{(h)}(U)$ is also a projective module over $O_Z(U)[G]$ and therefore that the last term of the previous sequence is equal to 0. This last exact sequence is precisely the one we require to complete the proof of the proposition.

Remark. We observe that in the case where E is a locally projective $O_Y[G]$ -module it follows that $\phi^*(E)$ is a projective $O_Z[G]$ -module and thus the proof of the proposition is complete without having to check the local projectivity of the modules $I^{(h)}, \Lambda^{(h)}$ and $\mathcal{G}^{(h)}$.

We deduce from Prop. 4.4 that for $0 \leq h \leq m-1$ the symmetric bundle obtained as the orthogonal sum of $(\phi^*(E)_{T^{(h)}})$, $(-1)^h \phi^*(q)_{T^{(h)}}$ and

$(\phi^*(E)_{T^{(h+1)}}, (-1)^{h+1}\phi^*(q)_{T^{(h+1)}})$ is metabolic with lagrangian $(\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)^\vee$. Hence, using the Main Lemma, we deduce that the product

$$w_t(\phi^*(E)_{T^{(h)}}, (-1)^h \phi^*(q)_{T^{(h)}}) w_t(\phi^*(E)_{T^{(h+1)}}, (-1)^{h+1} \phi^*(q)_{T^{(h+1)}})$$

is equal to $d_t((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)$. By taking the product of these equalities we then deduce that $w_t(\phi^*(E)_{T^{(0)}}, \phi^*(q)_{T^{(0)}})$ equals to

$$w_t(\phi^*(E)_{T^{(m)}}, (-1)^m \phi^*(q)_{T^{(m)}}) \prod_{0 \leq h \leq m-1} d_t((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G)^{(-1)^h}.$$

To deduce Thm. 0.8 from this equality we simply have to remember that $T^{(m)} = T$ by definition and that $(\phi^*(E)_{T^{(0)}}, \phi^*(q)_{T^{(0)}}) = \phi^*(E_{\rho, X}, q_{\rho, X})$ comes from Lemma 4.3.

We now establish the first part of Thm. 0.7 together with a preliminary version of the second part. Then in the next section we shall conclude the proof of the second part. As a consequence of the fact that for $0 \leq h \leq m$ the restrictions of the forms $(\phi^*(E)_{T^{(h)}}, \phi^*(q)_{T^{(h)}})$ to the generic fiber of Z coincide, we deduce that $w_1(\phi^*(E)_{T^{(h)}})$ and $c_1(\phi^*(E)_{T^{(h)}})$ do not depend on the choice of h . Moreover it also follows from Lemma 4.2 (c) that

$$w_2(\phi^*(E)_{T^h}) - w_2(\phi^*(E)_{T^{h+1}}) = c_1((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h)})^G) - c_1(\phi^*(E)_{T^{h+1}})$$

for $0 \leq h \leq m-1$. Therefore by adding these equalities we obtain that

$$w_2(\phi^*(E_{\rho, X})) - w_2(\phi^*(E)_T) = \delta(\phi^*(E), Z)$$

with

$$\delta(\phi^*(E), Z) = \sum_{1 \leq h \leq m} [c_1((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G) - c_1(\phi^*(E)_{T^{(h)}})] .$$

Using Thm. 0.4 to evaluate $w_k(\phi^*(E)_T) - w_k(\phi^*(E))$ we finally obtain that

$$w_1(\phi^*(E_{\rho, X})) = w_1(\phi^*(E)_T) = w_1(\phi^*(E)) + w_1(\phi^*(\rho))$$

and

$$\begin{aligned} w_2(\phi^*(E_{\rho, X})) &= w_2(\phi^*(E)) + w_1(\phi^*(E))w_1(\phi^*(\rho)) + w_2(\phi^*(\rho)) + \\ &+ sp_2(\phi^*(\rho)) + \delta(\phi^*(E), Z) . \end{aligned}$$

Remark. Since 2 is invertible in Z , we may associate to the squaring map on \mathbf{G}_m an exact Kummer sequence of étale sheaves of groups

$$0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{G}_m \rightarrow \mathbf{G}_m \rightarrow 0$$

and therefore an exact sequence of groups

$$0 \rightarrow \text{Pic}(Z)/2 \rightarrow H^2(Z, \mathbf{Z}/2\mathbf{Z}) \rightarrow H^2(Z, \mathbf{G}_m)_2 \rightarrow 0 .$$

The group $H^2(Z, \mathbf{G}_m)$ is known as the cohomological Brauer group of Z and is denoted by $Br'(Z)$, (see for instance [Mi], Chapter 4). It now follows from the definition of the first Chern class that $\delta(\phi^*(E), Z)$ belongs to $\text{Pic}(Z)/2$ (see below). Therefore by projecting our formula into $Br'(Z)$ we obtain a formula of Fröhlich-type ([F], Thms 2 and 3), in this group. It should be observed that in our case Z regular and integral—one believes that $Br'(Z)$ and the Brauer group of Z coincide. Furthermore, since Z is integral and regular, we have an exact sequence, [Es-K-V] 5.4

$$0 \rightarrow \text{Pic}(Z)/2 \rightarrow H^2(Z, \mathbf{Z}/2\mathbf{Z}) \rightarrow H^2(K, \mathbf{Z}/2\mathbf{Z}) \rightarrow 0 ,$$

where K denotes the function field of Z . Therefore, when we restrict our formulæ to the generic fiber of Z , the term $\delta(\phi^*(E), Z)$ disappears and so we obtain Fröhlich's original formula in the Galois cohomology group $H^2(K, \mathbf{Z}/2\mathbf{Z})$.

4.e Proof of Theorem 0.7

To complete the proof of Thm. 0.7 we shall often need to refer to [CNET1], Sect. 4. We first observe that, from the definition of the first Chern class, it follows that for any $1 \leq h \leq m$, the element of $H^2(Z, \mathbf{Z}/2\mathbf{Z})$

$$c_1((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G) - c_1(\phi^*(E)_{T^{(h)}})$$

is the image of the element $[\det(\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G] - [\det(\phi^*(E)_{T^{(h)}})]$ of $\text{Pic}(Z)$, where we denote by $[D]$ the class in $\text{Pic}(Z)$ of the divisor D of Z . Let $\alpha^{(h)} : \phi^*(E)_{T^{(h)}} \rightarrow (\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G$ be the inclusion map (see Prop. 4.4). We have the short exact sequence of O_Z -modules

$$0 \rightarrow \det(\phi^*(E)_{T^{(h)}}) \xrightarrow{\det(\alpha^{(h)})} \det((\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G) \rightarrow \text{coker}(\det(\alpha^{(h)})) \rightarrow 0 .$$

Therefore there exists a divisor $\Delta^{(h)}(E)$ such that

$$[\Delta^{(h)}(E)] = [\det(\phi^*(E) \otimes_{O_Z} \mathcal{G}^{(h-1)})^G] - [\det(\phi^*(E)_{T^{(h)}})] .$$

One of the main ingredients of the proof of the theorem is to provide an explicit description of each $\Delta^{(h)}(E)$ and thus of the sum $\Delta(E) = \sum_{h=1}^m \Delta^{(h)}(E)$ at least when restricted to each U_i of some étale covering $(U_i \rightarrow Z)$ of Z . Since $\Delta(E)$ has image $\delta(\phi^*(E), Z)$ in $H^2(Z, \mathbf{Z}/2\mathbf{Z})$, the theorem will follow from the congruence in $\text{Div}(Z)$

$$\Delta(E) \equiv \phi^*(R(\rho, X)) \pmod{2} .$$

This congruence will be deduced from the congruence mod 2 of the restrictions of both terms to each U_i of the covering. The final part of this section is devoted to the construction for any (closed) point of Z of an étale neighbourhood on which we can evaluate and compare as divisors the restrictions of $\Delta(E)$ and $\phi^*(R(\rho, X))$.

Before proceeding to these constructions and computations, we start by fixing some notation and by recalling the results of [CNET1] that we shall require. Let y be a point of Y of arbitrary dimension and $\sigma : S \rightarrow Y$ be an étale neighbourhood of y . For any scheme $v : V \rightarrow Y$ we write $V_S = V \times_Y S$, we define $O_V(S)$ as the ring of its global sections and denote by $v_S : V_S \rightarrow S$ and $\sigma_V : V_S \rightarrow V$ the projection maps. For a V -vector bundle \mathcal{F} we obtain the vector bundle $\mathcal{F}_S = \sigma_V^*(\mathcal{F})$ over V_S . We denote by $\mathcal{F}(S)$ the module of its global sections. Let now z be point of Z and $y = \phi(z)$. The étale neighbourhoods of z that we shall consider will always be of the form $\sigma_Z : Z_S \rightarrow Z$ where $\sigma : S \rightarrow Y$ is a well-chosen étale affine neighbourhood of y . As mentioned previously, the objects that we consider all have good functorial properties under base change. To be more precise: suppose that $\sigma_Z : Z_S \rightarrow Z$ is an étale neighbourhood of the type introduced above. For any $0 \leq h \leq m$ we can consider on the one hand $(T^{(h)})_S$ as defined previously and on the other hand $T_S^{(h)}$ as the normalisation of T'_S along the divisor $\sigma^*(b_1) \cup \dots \cup \sigma^*(b_h)$. From the fact that they coincide ([CNET1], Prop. 3.6), we deduce as in Thm. 0.6 that

$$(\phi^*(E)_{T^{(h)}})_S = ((\phi^*(E))_S)_{T_S^{(h)}} ,$$

and that we have an exact sequence of Z_S -vector bundles

$$0 \rightarrow (F \otimes_{O_{Z_S}} I_S^{(h)})^G \rightarrow F_{T_S^{(h)}} \oplus F_{T_S^{(h+1)}} \rightarrow (F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h)})^G \rightarrow 0 ,$$

where for simplicity we denote by F the O_{Z_S} -vector bundle $\phi^*(E)_S$. Therefore, using the fact that base-change commutes with taking determinants, for any $1 \leq h \leq m$, the restriction $\sigma_Z^*(\Delta^{(h)}(E))$ of $\Delta^{(h)}(E)$ to the étale neighbourhood Z_S of z will be obtained via the exact sequence

$$0 \rightarrow \det(F_{T_S^{(h)}}) \rightarrow \det(F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h-1)})^G \rightarrow \text{coker}(\det(\alpha_S^{(h)})) \rightarrow 0 ,$$

with $\alpha_S^{(h)} : F_{T_S^{(h)}} \rightarrow (F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h-1)})^G$ again being the inclusion map.

We now want to make precise the choice of the neighbourhood S which allows us to obtain explicit descriptions of the global sections of $F_{T_S^{(h)}}$ and $(F \otimes_{O_Z(S)} \mathcal{G}_S^{(h-1)})^G$ which lead themselves to determinantal computations.

We fix z a point of Z , we choose a point x of X whose image in Z is z and we write y for $\phi(z)$. For any $1 \leq h \leq m$, the inertia group of a generic point of $B_{h,k}$ only depends on h up to conjugacy, thus we may denote by I_h this group and by e_h the order of I_h . The cover X/Y being tame, I_h is cyclic and therefore will often be identified with $\mathbf{Z}/e_h\mathbf{Z}$. Let $I(x)$ be the inertia group of the point x . Then we know (see for instance Sect. 2 of [CNET1]) , that

$$I(x) \cong \prod_{\ell \in J(x)} I_\ell \cong \prod_{\ell \in J(x)} \mathbf{Z}/e_\ell\mathbf{Z} ,$$

where

$$J(x) = \{\ell \mid 1 \leq \ell \leq m, \exists k : x \in B_{\ell,k}\} .$$

After reordering if necessary we shall take $J(x) = \{1, 2, \dots, n\}$. For any such ℓ , we denote by χ_ℓ the character giving the action of I_ℓ on the cotangent space at the generic point of $B_{\ell,k}$. It follows from [CNET1], Lemma 2.3, that there exists an integral affine étale neighbourhood of y ,

$$S = \text{Spec}(A_y) ,$$

where A_y is an algebra containing a sequence a_1, a_2, \dots, a_n of regular parameters and enough roots of unity of order coprime with the residue characteristic of y and an isomorphism of schemes with G -action

$$X_S \cong \text{Spec}(O_X(S))$$

where

$$O_X(S) := \text{Map}_{I(x)}(G, B_x)$$

and

$$B_x := A_y[t_1, \dots, t_n] = A_y[T_1, \dots, T_n]/(T_1^{e_1} - a_1, \dots, T_n^{e_n} - a_n) .$$

Moreover, for $1 \leq \ell \leq n$, the action of I_ℓ on the image of T_ℓ in B_x that we denote by t_ℓ , is given by the character χ_ℓ . We may now obtain a description of Z_S and a description of T'_S considered as a scheme with G -action, from the above description of X_S . More precisely, if s denotes the cardinality of $G_2 \backslash G/I(x)$, we obtain that

$$Z_S = \text{Spec}(O_Z(S)) \quad \text{with} \quad O_Z(S) = \prod_{1 \leq j \leq s} B_{j,x} , \quad (4.5)$$

which is a product of s copies of $B_{j,x} = B_x$ and

$$T'_S = \text{Spec}(O_{T'}(S)) \quad \text{with} \quad O_{T'}(S) = \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} B_{j,x} \otimes_{A_y} B_x) ,$$

and $I(x)$ acting on the second factor of the tensor products. From now on the schemes S and Z_S given above will be the neighbourhoods of y and z that we shall consider.

We denote by $\mathcal{S}(x)$ the set of sequences $\alpha = (\alpha_\ell)$, $\ell \in J(x)$ where for each $\ell \in J(x)$ we have an integer α_ℓ such that $0 \leq \alpha_\ell < e_\ell$. For any $1 \leq h \leq m$, we consider the partition of $J = J(x)$ into

$$J'_h = J'_h(x) = \{\ell \in J : 1 \leq \ell \leq h\}$$

and

$$J''_h = J''_h(x) = \{\ell \in J : h+1 \leq \ell \leq m\} .$$

So for $h \geq n$, the set J''_h is empty. For $\alpha = (\alpha_\ell)$, we write

$$\partial^{(h)}(\alpha) := \left(\prod_{\ell \in J'_h} t_\ell^{-\alpha_\ell} \prod_{\ell \in J''_h} t_\ell^{-\epsilon_\ell} \right) \otimes \prod_{\ell \in J} t_\ell^{\alpha_\ell} ,$$

where $\epsilon_\ell = 0$ or e_ℓ depending on whether α_ℓ is strictly smaller or strictly larger than $e_\ell/2$. Then we define

$$D_j^{(h)}(\alpha) = B_{j,x} \partial^{(h)}(\alpha) \quad \text{and} \quad D_j^{(h)}(S) = \bigoplus_{\alpha \in \mathcal{S}(x)} D_j^{(h)}(\alpha) ,$$

and

$$\mathcal{G}_j^{(h-1)}(\alpha) = D_j^{(h-1)}(\alpha) + D_j^{(h)}(\alpha) \quad \text{and} \quad \mathcal{G}_j^{(h-1)}(S) = \bigoplus_{\alpha \in \mathcal{S}(x)} \mathcal{G}_j^{(h-1)}(\alpha) .$$

Let us denote by χ_x^α the character $\prod_{\ell \in J(x)} \chi_\ell^{\alpha_\ell}$ of $I(x)$; then we shall observe that the previous decompositions correspond to the decomposition according to the characters of $I(x)$. From now on, for any $I(x)$ -module M , we denote by $M(\alpha)$ the submodule of M on which the action of $I(x)$ is given by χ_x^α . It now follows from [CNET1], Sect. 3.e. that we have the isomorphisms of G -modules

$$I^{(h)}(S) \cong \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} I_j^{(h)}(S)) ,$$

$$\Lambda^{(h)}(S) \cong \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} D_j^{(h)}(S))$$

and

$$\mathcal{G}^{(h-1)}(S) \cong \text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} \mathcal{G}_j^{(h-1)}(S)) .$$

We start by deducing from these descriptions, that $I^{(h)}$, $\Lambda^{(h)}$ and $\mathcal{G}^{(h)}$ are locally projective $O_Z[G]$ -modules (recall that this was used in the proof of Prop. 4.4).

Lemma 4.6 *For any h , $1 \leq h \leq m$, the $O_Z[G]$ -modules $I^{(h)}$, $\Lambda^{(h)}$ and $\mathcal{G}^{(h)}$ are locally projective.*

PROOF. Let z be a point of Z , $y = \phi(z)$ and let $\sigma : S \rightarrow Y$ be the étale neighbourhood of y introduced above. Since σ is of finite type it is open. Let us denote by U the image $\sigma(S)$. We may assume that U is affine. In fact, since σ is étale and of finite type, it is smooth and quasi-finite, [Mi], I, Remark 3.25. Therefore it follows from Zariski's Main Theorem that it can be decomposed as the product of an open immersion and a finite map. Hence we are reduced to the case where $\sigma : S \rightarrow U$ is finite. It now follows from a theorem of Chevalley [EGA II], Thm. 6.7-1, that U is affine. Let us denote by V the affine open neighbourhood $\sigma^{-1}(U)$ of z and let us prove for instance that $I^{(h)}(V)$ is a projective $O_Z(V)[G]$ -module (the proof of the projectivity of $\Lambda^{(h)}(V)$ and $\mathcal{G}^{(h)}(V)$ will follow exactly the same lines). Let $\sigma_Z : Z_S \rightarrow Z$ be the morphism obtained from σ by base change. It is étale and moreover $\sigma_Z(Z_S) = V$. It follows that σ_Z induces a faithfully flat

morphism of affine schemes from Z_S onto V . Therefore it suffices to prove that $I^{(h)}(S)$ is a projective $O_Z(S)[G]$ -module. From the previous description of $I^{(h)}(S)$ we observe that this module is induced from the $O_Z(S)[I(x)]$ -module $\prod_{1 \leq j \leq s} I_j^{(h)}(S)$. From [CNET1], section 3.e., we know that this last module is finitely generated and free as an $O_Z(S)$ -module. Since the order of $I(x)$ is a unit in $O_Z(S)$, this is enough to conclude that it is projective as an $O_Z(S)[I(x)]$ -module and hence that $I^{(h)}(S)$ is a projective $O_Z(S)[G]$ -module. This completes the proof of the lemma.

We now return to the proof of Thm. 0.7. We recall that F has been defined at the beginning of this section as the O_{Z_S} -vector bundle $\phi(E)_S$. Denoting respectively by $F(S)$, $E(S)$ and $F_{T_S^{(h)}}(S)$ the global sections of F , $\varphi^*(E)$ and $F_{T_S^{(h)}}$, from the previous equalities we deduce that

$$(F(S) \otimes_{O_Z(S)} \mathcal{G}^{(h-1)}(S))^G \cong \left(\text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} (E(S) \otimes_{A_y} \mathcal{G}_j^{(h-1)}(S))) \right)^G$$

and

$$F_{T_S^{(h)}}(S) \cong \left(\text{Map}_{I(x)}(G, \prod_{1 \leq j \leq s} (E(S) \otimes_{A_y} D_j^{(h)}(S))) \right)^G.$$

For any $R[G]$ -module M and any subgroup H of G we recall that we denote by $\text{Map}_H(G, M)$ the set of maps $u : G \rightarrow M$ such that for any $x \in G$ and any $h \in H$, $u(hx) = hu(x)$, endowed with its structure of G -module defined by $gu : x \mapsto u(xg)$. It is now easy to check that for any such module M and any such subgroup H of G the map $f \mapsto f(1)$ induces an isomorphism of R -modules from $(\text{Map}_H(G, M))^G$ onto M^H . Therefore we conclude that we have the following isomorphisms

$$(F(S) \otimes_{O_Z(S)} \mathcal{G}^{(h-1)}(S))^G \cong \prod_{1 \leq j \leq s} (E(S) \otimes_{A_y} \mathcal{G}_j^{(h-1)}(S))^{I(x)}$$

and

$$F_{T_S^{(h)}}(S) \cong \prod_{1 \leq j \leq s} (E(S) \otimes_{A_y} D_j^{(h)}(S))^{I(x)}.$$

Denoting by $E_j(S)$ the tensor product $E(S) \otimes_{A_y} B_{j,x}$, we finally obtain that

$$(F(S) \otimes_{O_Z(S)} \mathcal{G}^{(h-1)}(S))^G \cong \prod_{1 \leq j \leq s} (E_j(S) \otimes_{B_{j,x}} \mathcal{G}_j^{(h-1)}(S))^{I(x)}$$

and

$$F_{T_S^{(h)}}(S) \cong \prod_{1 \leq j \leq s} (E_j(S) \otimes_{B_{j,x}} D_j^{(h)}(S))^{I(x)} .$$

We now want to consider the structure of E as a module when restricted to $I(x)$. To that end we consider E_y as an $O_{Y,y}[I(x)]$ -module. Since the order of $I(x)$ and the residue characteristic of y are coprime, E_y can be decomposed according to the characters of $I(x)$ as a direct sum

$$E_y = \sum_{\alpha \in \mathcal{S}(x)} E_y(\alpha) .$$

For any α , let us denote by $l_y(\alpha)$ the $O_{Y,y}$ -rank of $E_y(\alpha)$. Since E_y is an orthogonal representation of $I(x)$ we observe that for any $\alpha \in \mathcal{S}(x)$ it follows that $l_y(\alpha) = l_y(e - \alpha)$ where $e = (e_\ell)$, $\ell \in J(x)$. Therefore for any $1 \leq j \leq s$, we obtain a decomposition for $E_j(S)$, namely

$$E_j(S) = \sum_{\alpha \in \mathcal{S}(x)} E_j(S)(\alpha) ,$$

where $E_j(S)(\alpha)$ is a free $B_{j,x}$ -module of rank $l_y(\alpha)$. Hence we deduce that

$$(E_j(S) \otimes_{B_{j,x}} \mathcal{G}_j^{(h-1)}(S))^{I(x)} \cong \oplus_{\alpha \in \mathcal{S}(x)} (E_j(S)(\alpha) \otimes_{B_{j,x}} \mathcal{G}_j^{(h-1)}(S)(e - \alpha))$$

and

$$(E_j(S) \otimes_{B_{j,x}} D_j^{(h)}(S))^{I(x)} \cong \oplus_{\alpha \in \mathcal{S}(x)} (E_j(S)(\alpha) \otimes_{B_{j,x}} D_j^{(h)}(S)(e - \alpha)) .$$

We now return to the computations of determinants. For $h \notin J(x)$ and for any j and α we know from [CNET1], Prop. 3.14, that $D_j^{(h)}(S)(\alpha) = D_j^{(h-1)}(S)(\alpha)$ and so is equal to $\mathcal{G}_j^{(h-1)}(S)(\alpha)$. Therefore

$$\det(F_{T_S^{(h)}}) = \det(F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h-1)})^G .$$

Assuming now that $h \in J(x)$ we introduce the partition of $\mathcal{S}(x)$ into $\mathcal{S}_h(x)$ and $\mathcal{S}'_h(x)$ where $\mathcal{S}_h(x)$ (resp. $\mathcal{S}'_h(x)$) denotes the set of sequences α such that $e_h/2 < \alpha_h$ (resp. $\alpha_h < e_h/2$). We recall from [CNET1], Prop. 3.14 that $D_j^{(h)}(S)(\alpha) = (t_h^{(e_h - \alpha_h)} \otimes 1) \mathcal{G}_j^{(h-1)}(S)(\alpha)$ for $\alpha \in \mathcal{S}_h(x)$ and equal to $\mathcal{G}_j^{(h-1)}(S)(\alpha)$ otherwise. Hence we deduce from these equalities that, if $\alpha \in \mathcal{S}_h(x)$, then $(e - \alpha) \in \mathcal{S}'_h(x)$ and thus we have

$$\det(E_j(\alpha) \otimes_{B_{j,x}} \mathcal{G}_j^{(h-1)}(S)(e - \alpha)) = \det(E_j(\alpha) \otimes_{B_{j,x}} D_j^{(h)}(S)(e - \alpha)) .$$

If now $\alpha \in \mathcal{S}'_h(x)$, then $(e - \alpha) \in \mathcal{S}_h(x)$ and therefore

$$\det(E_j(\alpha) \otimes_{B_{j,x}} D_j^{(h)}(S)(e - \alpha)) = \det(E_j(\alpha) \otimes_{B_{j,x}} t_h^{\alpha_h} \mathcal{G}_j^{(h-1)}(S)(e - \alpha)) .$$

Using the fact that the $E_j(S)(\alpha)$ are free $B_{j,x}$ -modules of rank $l_y(\alpha)$ and that $D_j^{(h)}(S)(\alpha)$ and $\mathcal{G}_j^{(h-1)}(S)(\alpha)$ are free $B_{j,x}$ rank one modules ([CNET1], Prop. 3.14), we deduce from above that in this last case

$$\det(E_j(\alpha) \otimes_{B_{j,x}} D_j^{(h)}(S)(e - \alpha)) = \det(E_j(\alpha) \otimes_{B_{j,x}} \mathcal{G}_j^{(h-1)}(S)(e - \alpha)) t_h^{\alpha_h l_y(\alpha)} .$$

Thus for any $1 \leq j \leq s$,

$$\det((E_j(S) \otimes_{B_{j,x}} D_j^{(h)}(S))^{I(x)}) = \det((E_j(S) \otimes_{B_{j,x}} \mathcal{G}_j^{(h-1)}(S))^{I(x)}) t_h^{\sum_{\alpha \in \mathcal{S}'_h(x)} \alpha_h l_y(\alpha)} ,$$

and therefore from the previous equalities we may finally conclude that

$$\det(F_{T_S^{(h)}}) = \det((F \otimes_{O_{Z_S}} \mathcal{G}_S^{(h-1)})^G) \prod_{1 \leq j \leq s} t_h^{\sum_{\alpha \in \mathcal{S}'_h(x)} \alpha_h l_y(\alpha)} .$$

It follows from this equality that, for $h \in J(x)$, the restriction of $\Delta^{(h)}(E)$ to Z_S is defined as a Cartier divisor by

$$\Gamma^{(h)}(E) = \prod_{1 \leq j \leq s} t_h^{-\gamma^{(h)}(E)} ,$$

where we write $\gamma^{(h)}(E) = \sum_{\alpha \in \mathcal{S}'_h(x)} \alpha_h l_y(\alpha)$.

We now want to give an interpretation of $\gamma^{(h)}(E)$. We start by observing that, from its very definition,

$$\gamma^{(h)}(E) = \sum_{0 \leq k \leq e_h/2} k \sum_{\substack{\alpha \in \mathcal{S}_h(x) \\ \alpha_h = k}} l_y(\alpha) .$$

For any $1 \leq h \leq m$, let us denote by $d_k^{(h)}(E)$ the rank over O_{Y,ξ_h} of the χ_h^k component of E_{ξ_h} , when considered as an I_h -module. When $h \in J(x)$, since O_{Y,ξ_h} contains $O_{Y,y}$, we can write E_{ξ_h} as the tensor product $E_y \otimes_{O_{Y,y}} O_{Y,\xi_h}$. We then deduce from this equality the following decomposition of E_{ξ_h} into a direct sum of $I(x)$ -modules:

$$E_{\xi_h} = \oplus_{\alpha \in \mathcal{S}(x)} ((E_y)(\alpha) \otimes_{O_{Y,y}} O_{Y,\xi_h}) .$$

This therefore implies that the χ_h^k -component of E_{ξ_h} , when considered as an I_h -module, is the direct sum of the $E_y(\alpha) \otimes_{O_{Y,y}} O_{Y,\xi_h}$ when α runs through the elements of $\mathcal{S}_h(x)$ such that $\alpha_h = k$. It then follows from this decomposition that, for $0 \leq k < e_h/2$,

$$d_k^{(h)}(E) = \sum_{\substack{\alpha \in \mathcal{S}_h(x) \\ \alpha_h = k}} l_y(\alpha) .$$

We have then proved that for $h \in J(x)$

$$\gamma^{(h)}(E) = d^{(h)}(E) ,$$

where $d^{(h)}(E)$ has been defined in the introduction as the sum $\sum_{k=0}^{e_h/2} k d_k^{(h)}(E)$.

We now come back to the function $\Gamma^{(h)}(E)$ which defines $\Delta^{(h)}(E)$. Writing $N = \prod_{h \in J(x)} e_h$ and, for $h \in J(x)$, $N_h = N/e_h$ and using that $t_h^{e_h} = a_h$ for any such h , we obtain that

$$\Gamma^{(h)}(E)^N = a_h^{-N_h d^{(h)}(E)} .$$

Since by hypothesis the ramification indices are odd we conclude that the restriction of $\Delta(E)$ to Z_S , namely $\sigma_Z^*(\Delta(E))$, is defined as a Cartier divisor by the function

$$\Gamma(E) = \prod_{h \in J(x)} \Gamma^{(h)}(E) \equiv \prod_{h \in J(x)} a_h^{d^{(h)}(E)} \pmod{2} .$$

Let us now consider the ramification divisor

$$R(\rho, X) = \sum_{1 \leq h \leq m} d^{(h)}(E) b_h$$

defined in the introduction. We write U for the image of σ . We assume that S has been chosen sufficiently small such that the irreducible components that U intersects are precisely those containing y , namely $\{b_h, h \in J(x)\}$. Since $\sigma : S \rightarrow U$ is étale, it follows that

$$\sigma^*(R(\rho, X)) = \sum_{h \in J(x)} d^h(E) \sum_{\varphi(\eta) = \xi_h} \{\bar{\eta}\} ,$$

where the η 's are points of S of codimension 1 over ξ_h . Therefore, since a_h is a local equation of b_h for any such h , we obtain that $\sigma^*(R(\rho, X))$ and therefore

$\phi_S^*(\sigma^*(R(\rho, X)))$ is defined by the function $\prod_{h \in J(x)} a_h^{d^{(h)}(E)}$. Using the equality $\phi_S^*(\sigma^*(R(\rho, X)(E))) = \sigma_Z^*(\phi^*(R(\rho, X)))$ and the congruence satisfied by the function $\Gamma(E)$ we conclude that the restrictions of $\phi^*(R(\rho, X))$ and $\Delta(E)$ to Z_S are indeed congruent mod 2 as required.

Example. Our final goal is to compute the divisor $R(\rho, X)$ in a special case considered in Sect. 2.e. We keep the hypotheses and the notations of Example 2.6. So we consider the symmetric bundle (E, q) where $E = O_Y[G/H]$ and q is the symmetric form which has the cosets $\{\bar{a} = aH\}$ as an orthonormal basis; ρ is the tame orthogonal representation of G induced by permuting the \bar{a} .

By the above work the computation of this divisor $R(\rho, X)$ reduces to evaluating the integers $d_k^{(h)}(E)$ for any $1 \leq h \leq m$ and $1 \leq k \leq e_h/2$. We now fix such an h , we choose once for all a codimension one point ξ_h'' of X above ξ_h and we assume for simplicity that O_{Y, ξ_h} contains the values of the character χ_h . We let I_h (resp. Δ_h) denote the inertia group (resp. decomposition group) of ξ_h'' . Since there is no risk of ambiguity we make no further mention from now of the dependance upon h of the objects we consider and therefore we will write I for I_h , Δ for Δ_h , ξ for ξ_h , etc.

If $G = \cup_{1 \leq i \leq r} \Delta \gamma_i H$ is a double coset decomposition of G then, by standard theory (see for instance [FT], Chap. 8, Sect. 7) we have an isomorphism of left $O_{Y, \xi}[\Delta]$ -modules

$$E_\xi \cong \oplus_{1 \leq i \leq r} O_{Y, \xi}[\Delta / (\gamma_i H \cap \Delta)] ,$$

where for $\gamma \in G$ we write $\gamma H = \gamma H \gamma^{-1}$. We observe that the above double cosets parametrise the codimension one points of $V = X/H$ above ξ by the rule $\gamma_i \mapsto \lambda(\xi^{\gamma_i})$ (we recall that G acts on the right on X). Then $\Delta_i = H \cap \gamma_i^{-1} \Delta$ (resp. $I_i = H \cap \gamma_i^{-1} I$) is the decomposition group (resp. the inertia group) of ξ''^{γ_i} over V . We recall that we write e (resp. f) for the ramification index (resp. the residue class extension degree) of ξ'' over Y . Let us write e'_i (resp. f'_i) for the ramification index (resp. the residue class extension degree) of ξ''^{γ_i} over V , thus the codimension one point on V corresponding to γ_i , namely $\lambda(\xi^{\gamma_i})$, has ramification $e_i = e e_i'^{-1}$ and residue class extension degree $f_i = f f_i'^{-1}$. It follows from the above isomorphism that in order to decompose E_ξ as a direct sum of $O_{Y, \xi}[I]$ -modules we have to decompose each $O_{Y, \xi}[\Delta / (\gamma_i H \cap \Delta)]$. With this in mind we consider the double cosets decomposition of $I \backslash \Delta / (\gamma_i H \cap \Delta)$. Using the fact that I is a normal subgroup

of Δ we observe that each component of the direct sum decomposition is isomorphic to $O_{Y,\xi}[I/(\gamma_i H \cap I)]$ and that the number of components, namely the number of the sets of double cosets, is equal to f_i . Therefore we have proved that there is an isomorphism of $O_{Y,\xi}[I]$ -modules

$$E_\xi \cong \bigoplus_{1 \leq i \leq r} \bigoplus_{1 \leq j_i \leq f_i} O_{Y,\xi}[I/(\gamma_i H \cap I)] .$$

For any $O_{Y,\xi}[I]$ -module M and for any integer $1 \leq k \leq e_h/2$, let us denote by $M(k)$ the χ^k -component of M . For each k we deduce from above the following decomposition

$$E_\xi(k) \cong \bigoplus_{1 \leq i \leq r} \bigoplus_{1 \leq j_i \leq f_i} O_{Y,\xi}[I/(\gamma_i H \cap I)](k) .$$

Therefore the computation of $d_k(E)$ reduces to the computation of the rank $r_{\gamma_i}(k)$ over $O_{Y,\xi}$ of each $O_{Y,\xi}[I/(\gamma_i H \cap I)](k)$. It is clear that this module is generated as an $O_{Y,\xi}$ -module by the set $\{\epsilon_{\chi^k} \bar{a}\}$ when \bar{a} runs through $I/(\gamma_i H \cap I)$ and where ϵ_{χ^k} is the idempotent of $O_{Y,\xi}[I]$ associated to the character χ^k , namely $e^{-1} \sum_{u \in I} \chi^k(u) u^{-1}$. For any \bar{a} we have the equality

$$\epsilon_{\chi^k} \bar{a} = e^{-1} \sum_{u \in I} \chi^k(u) u^{-1} \bar{a} .$$

We now observe that $u^{-1} \bar{a} = \bar{a}$ is equivalent to $u \in \gamma_i H \cap I$. Hence, if for any $1 \leq v \leq s$ we denote by u_v , a full set of representatives of $I/(\gamma_i H \cap I)$, then we obtain

$$\epsilon_{\chi^k} \bar{a} = e^{-1} \sum_{t \in (\gamma_i H \cap I)} \chi^k(t) \sum_{1 \leq v \leq s} \chi_h^k(u_v) u_v^{-1} \bar{a} .$$

It follows from the definitions that the integer s is equal to the ramification index e_i . It is now clear from this last equality that $O_{Y,\xi}[I/(\gamma_i H \cap I)](k)$ is different from 0 if and only if χ^k is trivial when restricted to $\gamma_i H \cap I$, namely when k belongs to the set of integers $\{e'_i t, 1 \leq t \leq e_i/2\}$. For any t , $1 \leq t \leq e_i/2$, we obtain that

$$\epsilon_{\chi^{e'_i t}} \bar{a} = e_i^{-1} \sum_{u \in I/(\gamma_i H \cap I)} \chi^{e'_i t}(u) u \bar{a} .$$

Therefore, for $1 \leq t \leq e_i/2$, we deduce that $O_{Y,\xi}[I/(\gamma_i H \cap I)](e'_i t)$ is a free, rank 1, $O_{Y,\xi}$ -module with $\{\epsilon_{\chi^{e'_i t}} \bar{1}\}$ as a free basis. Hence we have proved that $r_{\gamma_i}(k) = r_i(k) = 1$ if e'_i divides k and 0 otherwise. Therefore we have:

$$d(E) = \sum_{0 \leq k \leq e/2} k d_k(E) = \sum_{0 \leq k \leq e/2} k \sum_{1 \leq i \leq r} f_i k r_i(k) .$$

and thus

$$d(E) = \sum_{1 \leq i \leq r} e'_i f_i \sum_{0 \leq k \leq e_i/2} k = \sum_{1 \leq i \leq r} e'_i f_i \frac{(e_i^2 - 1)}{8} .$$

Since the ramification indices are odd we conclude that in this case

$$R(\rho, X) \equiv \sum_h \left(\sum_{\eta_h \rightarrow \xi_h} f(\eta_h) \frac{(e(\eta_h)^2 - 1)}{8} \right) b_h \pmod{2} ,$$

where η_h ranges over codimension points on V above ξ_h . The right hand side of this congruence is precisely the divisor obtained in [CNET1] and the divisor obtained by Serre in [Se2].

5 Appendix: Hasse-Witt invariants of symmetric complexes

The aim of this note is to indicate how one can define Hasse-Witt invariants for symmetric complexes using ideas of Saito and Walter (see [S2] and [Wa] for details). These invariants have been used in [CNET2] to give a more direct approach to the comparison result Thm. 0.1 in [CNET1]. They also have been used by Saito to formulate a conjecture relating invariants of forms coming from ℓ -adic and de Rham cohomology of a projective smooth scheme (see [S2]).

It should be clear that we do not make any claim to originality. The construction of the invariants is due to Walter and Saito.

5.a Symmetric complexes

We have defined a symmetric bundle (E, ϕ) over a noetherian $\mathbf{Z}[\frac{1}{2}]$ -scheme Y to be a vector bundle E over Y equipped with a symmetric isomorphism ϕ between E and its Y -dual E^\vee , that is to say $\phi : E \cong E^\vee$ and ϕ is equal to its transpose ϕ^\vee after identifying E with $E^{\vee\vee}$. A *symmetric complex* on Y is a symmetric object (P_\bullet, ϕ) of the derived category $\mathcal{D}^b(Y)$ of bounded complexes of Y -vector bundles. This is a triangulated category with a duality which extends \vee : namely the localisation of the functor which sends a complex P_\bullet to the dual complex P_\bullet^\vee . Symmetry is defined using the natural identification of a complex with its double dual (see Sect. 2 of [Ba1]). A symmetric bundle may therefore be viewed as a symmetric complex concentrated in degree zero. One can define a metabolic object in any triangulated category. In the case of $\mathcal{D}^b(Y)$ we say that (P_\bullet, ϕ) is *metabolic* with lagrangian L_\bullet if there is a distinguished triangle

$$L_\bullet \xrightarrow{i} P_\bullet \xrightarrow{i^\vee \circ \phi} L_\bullet^\vee \xrightarrow{w} TL_\bullet$$

in the derived category with the duality condition that $T(w^\vee) = w$. One can then define a Witt group for $\mathcal{D}^b(Y)$, and an object in $\mathcal{D}^b(Y)$ is metabolic if and only if it is zero in this Witt group (see Thm. 3.5 in [Ba1]).

Proposition 5.1 *For any symmetric complex (P_\bullet, ϕ) there exists a symmetric bundle (E', γ) such that the orthogonal sum $(P_\bullet, \phi) \perp (E', \gamma)$ is metabolic with lagrangian given by $P_{<0} \oplus E'$.*

We briefly sketch a proof of the proposition following indications of Walter and using the results in [Wa]. (Note that this reference contains an alternative approach to Balmer's theorem which identifies the Witt group of Y with the Witt group of $\mathcal{D}^b(Y)$ (see [Ba2]).)

PROOF. The starting point is the existence of a factorisation of $\phi : P_\bullet \rightarrow P_\bullet^\vee$ as:

$$\begin{array}{ccccccc}
P_\bullet & \cdots & \rightarrow & P_1 & \rightarrow & P_0 & \xrightarrow{d_0} P_{-1} \rightarrow \cdots \\
\downarrow f & & & \downarrow & & \downarrow \pi_1 & \parallel \\
Q_\bullet & \cdots & \rightarrow & P_{-1}^\vee & \xrightarrow{\pi_0} & E'' & \rightarrow P_{-1} \rightarrow \cdots \\
\downarrow \psi & & & \parallel & & \downarrow \lambda & \parallel \\
Q_\bullet^\vee & \cdots & \rightarrow & P_{-1}^\vee & \rightarrow & E''^\vee & \rightarrow P_{-1} \rightarrow \cdots \\
\downarrow f^\vee & & & \parallel & & \downarrow & \downarrow \\
P_\bullet^\vee & \cdots & \rightarrow & P_{-1}^\vee & \xrightarrow{d_0^\vee} & P_0^\vee & \rightarrow P_1^\vee \rightarrow \cdots
\end{array} \tag{5.1}$$

where (E'', λ) is a symmetric bundle defined using the acyclicity of the mapping cone of ϕ . Namely, we factor

$$P_{-1}^\vee \oplus P_0 \begin{pmatrix} d_0^\vee & \phi_0 \\ 0 & -d_0 \end{pmatrix} \rightarrow P_0^\vee \oplus P_{-1}$$

into a composition of an essential epi π with an essential mono σ

$$P_{-1}^\vee \oplus P_0 \xrightarrow{\pi} E'' \xrightarrow{\sigma} P_0^\vee \oplus P_{-1}$$

where $\pi = (\pi_0, \pi_1)$, $\sigma = \begin{pmatrix} \sigma_0 \\ \sigma_1 \end{pmatrix}$ (see Lemma 4.2 *loc. cit.*). This then defines

E'' and provides a factorisation of ϕ as $P_\bullet \xrightarrow{f} Q_\bullet \xrightarrow{f^\vee \circ \psi} P_\bullet^\vee$, so that there is a unique $\lambda : E'' \rightarrow E''^\vee$ with the expected properties (see Prop. 5.3 *loc. cit.*). Let $L_\bullet : \cdots \rightarrow 0 \rightarrow 0 \rightarrow P_{-1} \rightarrow \cdots$ be the truncation of P_\bullet in degrees less than zero, then from (5.1) we obtain

$$L_\bullet \xrightarrow{u} Q_\bullet \xrightarrow{\psi} Q_\bullet^\vee \xrightarrow{u^\vee} L_\bullet^\vee.$$

The complex L_\bullet is totally isotropic in Q_\bullet , i.e. $u^\vee \circ \psi \circ u = 0$. *A priori* this holds in $\mathcal{D}^b(Y)$, but Prop. 3.1 in [Wa] shows that, after possibly changing (Q_\bullet, ψ) to $(\tilde{Q}_\bullet, \tilde{\psi})$ by an isomorphism in $\mathcal{D}^b(Y)$, we can lift all maps to maps of chain complexes such that the resulting chain map lifting u^\vee is a split epi.

If L_{\bullet}^{\perp} denotes the kernel of this map, then one can consider the subquotient $L_{\bullet}^{\perp}/L_{\bullet}$, on which $\tilde{\psi}$ defines a symmetric quasi-isomorphism q . In fact, by Lemma 3.2 of *loc. cit.*, $[L_{\bullet}^{\perp}/L_{\bullet}, q] = [Q_{\bullet}, \psi]$ in the derived Witt group, and hence

$$[E'', \lambda] = [L_{\bullet}^{\perp}/L_{\bullet}, q] = [Q_{\bullet}, \psi] = [P_{\bullet}, f^{\vee} \psi f] = [P_{\bullet}, \phi]$$

and the proposition follows by letting $E' = E''$ and $\gamma = -\lambda$.

5.b Definition of the invariants

We are now in a position to define the Hasse-Witt invariant of a symmetric complex. Recall that the total Hasse-Witt invariant of a symmetric bundle $E = (E, \phi)$ over Y is

$$w_t(E) = \sum_{i \geq 0} w_i(E) t^i$$

where $w_i(E)$ belongs to $H^i(Y) := H^i(Y_{et}, \mathbf{Z}/2\mathbf{Z})$. As the main lemma of Sect. 4 shows, this invariant does not in general vanish on metabolic bundles. Recall that if E is a metabolic bundle with lagrangian V of rank n and if $c_i(V)$ is the i th Chern class in $H^{2i}(Y)$, then

$$w_t(E) = d_t(V) := \sum_{i=0}^n (1 + (-1)^i t)^{n-i} c_i(V) t^{2i}.$$

We extend the definition of $d_t(-)$ to complexes by multiplicativity, so that $d_t(L_{\bullet}) = \prod_i d_t(L_i)^{(-1)^i}$. Guided by the above equality, forcing additivity and using the notation of the proposition, we then recoup Saito's definition of Hasse-Witt classes for symmetric complexes by putting

$$w_t(P_{\bullet}, \phi) := w_t(E', -\gamma) d_t(P_{<0}) .$$

We note that these Hasse-Witt classes have all the standard properties of characteristic classes. In particular they are natural with respect to pull-back and satisfy the Whitney sum formula on sums of symmetric complexes; furthermore the main lemma extends to metabolic complexes (see [S2]).

References

- [Ba1] Balmer P.: *Triangular Witt groups. Part I: the 12-term localisation exact sequences*. K-theory, **19**(2000), 311-363.
- [Ba2] Balmer P.: *Triangular Witt groups. Part II: from usual to derived*. Math. Z. , **236**(2001), 351-382.
- [C-E] Chinburg, T., Erez, B.: *Equivariant Euler-Poincaré characteristics and tameness*. Journées Arithmétiques, 1991. Astérisque **209**(1992), 13, 179–194.
- [CEPT1] Chinburg, T., Erez, B., Pappas, G., Taylor, M.J.: *Tame actions of group schemes: integrals and slices*. Duke Math. J. **82**(1996), no. 2, 269–308.
- [CEPT2] Chinburg, T., Erez, B., Pappas, G., Taylor, M.J.: *ϵ -constants and the Galois structure of de Rham cohomology*. Ann. of Math. (2)**146**(1997), no. 2, 411–447.
- [CNET1] Cassou-Noguès, Ph., Erez, B., Taylor, M.J.: *Invariants of a quadratic form attached to a tame covering of schemes*. J. Th. des Nombres de Bordeaux **12**(2000), 597-660.
- [CNET2] Cassou-Noguès, Ph., Erez, B., Taylor, M.J.: *Hasse-Witt invariants of symmetric complexes: an example from geometry*. C. R. Acad. Sci. Paris Sér. I Math. **334**(2002), 839-842.
- [De] Deligne, P.: *Les constantes locales de l'équation fonctionnelle de la fonction L d'Artin d'une représentation orthogonale*. Invent. Math. **35**(1976), 299-316.
- [Dz] Delzant, A.: *Définition des classes de Stiefel-Whitney d'un module quadratique sur un corps de caractéristique différente de 2*. C. R. Acad. Sci. Paris **255**(1962), 1366–1368.
- [E] Erez, B.: *Geometric trends in Galois module theory.*, 116-145, in Galois representations and arithmetic algebraic geometry, eds A.J. Scholl and R.L. Taylor, LMS Lectures Notes **254**, Cambridge University Press, Cambridge, 1998.

- [Es-K-V] Esnault, H., Kahn, B., Viehweg, E.: *Coverings with odd ramification and Stiefel-Whitney classes*. J. reine angew. Math. **441**(1993), 145–188.
- [F] Fröhlich, A.: *Orthogonal representations of Galois groups, Stiefel-Whitney classes and Hasse-Witt invariants*. J. reine angew. Math. **360**(1985), 84–123.
- [FQ] Fröhlich, A., Queyrut, J.: *On the functional equation of the Artin L-function for characters of real representations*. Invent. Math. **20**(1973), 125–138.
- [FT] Fröhlich, A., Taylor, M.J.: *Algebraic number theory*. Cambridge studies in advanced Mathematics **27**. Cambridge University Press, 1991.
- [Gl] Glass, D.: *Invariants associated to orthogonal ϵ -constants*. Columbia, preprint 2002.
- [Gr] Grothendieck, A.: *Classes de Chern et représentations linéaires des groupes discrets*. Dix Exposés sur la Cohomologie des Schémas pp. 215–305, North-Holland, Amsterdam; Masson, Paris, 1968.
- [Gr-M] Grothendieck, A., Murre, J.P.: *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme*. Lect. Notes in Math., **208**. Springer-Verlag, Berlin-New York, 1971.
- [EGA II] Grothendieck, A., Dieudonné, J.: *Eléments de géométrie algébrique II. Etude globale élémentaire de quelques classes de morphismes*. Inst. Hautes Etudes Sci. Publ. Math. **8**(1961).
- [J1] Jardine, J. F.: *Universal Hasse-Witt classes*. Algebraic K-theory and algebraic number theory, 83–100, Contemp. Math., **83**, Amer. Math. Soc., Providence, RI, 1989.
- [J2] Jardine, J. F.: *Higher spinor classes*. Mem. Amer. Math. Soc. **528**(1994).

- [K] Kahn, B.: *Equivariant Stiefel-Whitney classes*. J. Pure Appl. Algebra **97**(1994), no. 2, 163–188.
- [Kne] Knebusch, M.: *Symmetric bilinear forms over algebraic varieties*, Conference on Quadratic Forms—1976 , pp. 103–283 ed. G. Orzech. Queen’s Papers in Pure and Appl. Math., **46**, Queen’s Univ., Kingston, Ont., 1977.
- [Knu] Knus, M-A.: Quadratic and Hermitian forms over rings. Grundlehren der Math. Wiss. **294**, Springer-Verlag, Berlin, 1991.
- [McL] MacLane, S.: Homology. Grundlehren der Math. Wiss **114**, Springer-Verlag, Berlin, 1975.
- [Mi] Milne, J.S.: Etale cohomology. Princeton Mathematical Series, **33**. Princeton University Press, Princeton, N.J., 1980.
- [O] Ojanguren, M.: The Witt group and the problem of Lüroth, Dottorato di Ricerca in Mat., Univ. di Pisa, ETS Editrice, Pisa, 1990.
- [S1] Saito, T.: *The sign of the functional equation of the L-function of an orthogonal motive*. Invent. Math. **120**(1995), 119-142.
- [S2] Saito, T.: *Note on Stiefel-Whitney class of ℓ -adic cohomology*. Preprint, University of Tokyo, 1998.
- [SGA4] Théorie des topos et cohomologie étale des schémas. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. Lecture Notes in Math., **269**, **270**, **305**. Springer-Verlag, Berlin-New York, 1972-73.
- [Se1] Serre, J-P.: *L’invariant de Witt de la forme $\text{Tr}(x^2)$* . Comment. Math. Helv. **59**(1984), no. 4, 651–676.
- [Se2] Serre, J-P.: *Revêtements à ramification impaire et thêta-caractéristiques*. C. R. Acad. Sci. Paris Sér. I Math. **311**(1990), no. 9, 547-552.

- [Se3] Serre, J-P.: Cohomologie galoisienne. Cinquième édition, révisée et complétée. Lecture Notes in Mathematics **5**. Springer-Verlag, Berlin, 1994.
- [Se4] Serre, J-P.: Corps locaux. Hermann, 1968.
- [Sn] Snaith, V.P.: *Stiefel-Whitney classes of bilinear forms—a formula of Serre*. Can. Bull. Math. **28**(1985), no. 2, 218-222.
- [Wa] Walter, C.: *Obstructions to the existence of symmetric resolutions*. Preprint, Nice, 2001 (see also *Grothendieck and Witt groups of triangulated categories*, Preprint, Nice, 2003).

AUTHORS' ADDRESSES:

Ph. C-N. & B. E.: Laboratoire “Théorie des nombres et algorithmique arithmétique” (UMR CNRS 5465), Institut de Mathématiques de Bordeaux (FR CNRS 2254), Université Bordeaux 1, 351, cours de la Libération, F-33405 Talence, France

e-mail : phcassou@math.u-bordeaux.fr erez@math.u-bordeaux.fr

M.J. T.: Department of Mathematics, U.M.I.S.T., P.O. Box 88, GB-Manchester M 60 1 QD, England, U.K.

e-mail : martin.taylor@umist.ac.uk